Indian J. pure appl. Math., **20** (11) : 1119-1125, November 1989

INVARIANT SUBMANIFOLDS IN A CONFORMAL *K*-CONTACT RIEMANNIAN MANIFOLD

B. RAVI AND C. S. BAGEWADI

Department of Studies in Mathematics, Kuvempu University, B. R. Project 577115
Dist. Shimoga, Karnataka State

(*Received* 14 *June* 1988; *after revision* 29 *March* 1989; *accepted* 8 *May* 1989)

Kon[1] gave necessary conditions for an invariant submanifold M of a K-contact Riemannian manifold $\bar{M}$ to be minimal and gave the necessary and sufficient conditions for M to be totally geodesic. Further Endo[2] also has shown that an invariant submanifold M of a K-contact Riemannian manifold $\bar{M}$ is minimal. Here we give some conditions for the invariant submanifold of a conformal K-contact Riemannian manifold to be minimal and totally geodesic.

1. PRELIMINARIES

A conformal K-contact Riemannian manifold $\bar{M}$ is defined[3] as an almost Sasakian manifold with structure tensors $\bar{\phi}$, $\bar{\xi}$, $\bar{\eta}$, $\bar{g}$ in which the associated vector field $\bar{\xi}$ is a conformal killing vector field i. e. in which

$$(\bar{\nabla}_{\bar{X}}\bar{\eta})(\bar{Y}) + (\bar{\nabla}_{\bar{Y}}\bar{\eta})(\bar{X}) = 2\alpha\, \bar{g}(\bar{X}, \bar{Y})$$

where $\bar{\nabla}$ is the Riemannian connection, $\bar{X}$, $\bar{Y}$ are vector fields on $\bar{M}$ and α is a scalar. If, in particular, $\alpha = 0$, then the manifold is a K-contact manifold[4].

For a conformal K-contact Riemannian manifold $\bar{M}$, we have[3]

$$\bar{\phi}\,\bar{\xi} = 0,\quad \bar{\eta}(\bar{\xi}) = 1,\quad \bar{\phi}^2 \bar{X} = -\bar{X} + \bar{\eta}(\bar{X})\,\bar{\xi} \qquad \ldots(1.1)$$

$$\bar{g}(\bar{\phi}\bar{X}, \bar{\phi}\bar{Y}) = \bar{g}(\bar{X}, \bar{Y}) - \bar{\eta}(\bar{X})\,\bar{\eta}(\bar{Y}),\quad \bar{\eta}(\bar{X}) = \bar{g}(\bar{\xi}, \bar{X}) \qquad \ldots(1.2)$$

for any vector fields $\bar{X}$ and $\bar{Y}$ on $\bar{M}$.

It is known[3] that in a conformal K-contact Riemannian manifold, the following formulas hold :

$$\bar{\Phi}(\bar{X}, \bar{Y}) = \alpha\,\bar{g}(\bar{X}, \bar{Y}) - (\bar{\nabla}_{\bar{Y}}\bar{\eta})(\bar{X}) = (\bar{\nabla}_{\bar{X}}\bar{\eta})(\bar{X}) - \alpha\,\bar{g}(\bar{X}, \bar{Y}) \qquad \ldots(1.3)$$

$$\bar{K}(\bar{X}, \bar{Y}, \bar{Z}, \bar{\xi}) + \bar{K}(\bar{\phi}\bar{X}, \bar{\phi}\bar{Y}, \bar{Z}, \bar{\xi}) = \bar{\Phi}(\bar{X}, \bar{Z})(\bar{\phi}\bar{Y}\alpha)$$
$$- \bar{\Phi}(\bar{Y}, \bar{Z})(\bar{\phi}\bar{X}\alpha) + \bar{g}(\bar{X}, \bar{Z})(\bar{Y}\alpha) - \bar{g}(\bar{Y}, \bar{Z})(\bar{X}\alpha)$$
$$+ \bar{\eta}(\bar{Y})\bar{\eta}(\bar{\nabla}_{\bar{Z}}\bar{\phi}\bar{X}) - \bar{\eta}(\bar{X})\bar{\eta}(\bar{\nabla}_{\bar{Z}}\bar{\phi}\bar{Y}) \quad \ldots(1.4)$$

$$\bar{g}(\bar{X}, \bar{Y}) + \alpha\bar{\Phi}(\bar{X}, \bar{Y}) + \bar{\eta}(\bar{\nabla}_{\bar{Y}}\bar{\phi}\bar{X}) = \bar{\eta}(\bar{X})\bar{\eta}(\bar{Y}) \quad \ldots(1.5)$$

where $\bar{K}$ is the Riemannian curvature tensor on $\bar{M}$ and $\bar{\Phi}(\bar{X}, \bar{Y}) = \bar{g}(\bar{\phi}\bar{X}, \bar{Y})$.

By virtue of the above relations, we have

$$\bar{\nabla}_{\bar{X}}\bar{\xi} = \bar{\phi}\bar{X} + \alpha\bar{X} \quad \ldots(1.6)$$

$$\bar{K}(\bar{X}, \bar{\xi})\bar{Y} = (\bar{\nabla}_{\bar{X}}\bar{\phi})\bar{Y} + (\bar{\xi}\alpha)\bar{g}(\bar{X}, Y)\bar{\xi} - (\bar{X}\alpha)\bar{Y} \quad \ldots(1.7)$$

$$\bar{K}(\bar{X}, \bar{\xi})\bar{\xi} = \bar{X} - \bar{\eta}(\bar{X})\bar{\xi} + (\bar{\xi}\alpha)\bar{\eta}(\bar{X})\bar{\xi} - (\bar{X}\alpha)\bar{\xi}. \quad \ldots(1.8)$$

Let M be an $(2m + 1)$ dimensional $(n > m)$ manifold imbedded in $\bar{M}$. The induced metric g of M is given by $g(X, Y) = \bar{g}(\bar{X}, \bar{Y})$ for any vector fields X, Y on M. Let $T_x(M)$ and $T_x(M)^{\perp}$ denote that tangent and normal bundles of M and $x \in M$. Let ∇_X denote the Riemannian connection on M determined by the induced metric g and K denote the Riemannian curvature tensor of M. Then Gauss-Weingarten formula is given by

$$\bar{\nabla}_X Y = \nabla_X Y + B(X, Y), \quad \bar{\nabla}_X N = -A_N(X) + D_X N \quad \ldots(1.9)$$

for any vector fields X, Y tangent to M and any vector fleld N normal to M, where D is the operator of covariant differentiation with respect to the linear connection induced in the normal bundle $T_X(M)^{\perp}$. Both A and B are called the second fundamental forms of M and they satisfy $\bar{g}(B(X, Y), N) = g(A_N(X, (Y)$.

A submanifold M of $\bar{M}$ is said to be invariant if $\bar{\xi}$ tangent to M everywhere on M and $\bar{\phi}X$ is tangent to M for any tangent vector X to M. An invariant submanifold M has the induced structure tensors (ϕ, ξ, η, g).

2. Invariant Submanifolds in Conformal K-contact Riemannian Manifold

Let $\bar{M}$ be a $(2n + 1)$ dimensional conformal K-contact Riemannian manifold and M a $(2m + 1)$ dimensional $(n > m)$ manifold imbedded in $\bar{M}$.

For the second fundamental form B of an invariant submanifold M of a conformal K-contact Riemannian manifold, we define its covariant derivative $(\tilde{\nabla}_X B)$ by

$$(\tilde{\nabla}_X B)(Y, Z) = D_X (B(Y, Z)) - B(\nabla_X Y, Z) - B(Y, \nabla_X Z). \quad \ldots(2.1)$$

$X, Y, Z \in X(M)$— the set of all differentiable vector fields on M.

Then by (1.9), we obtain

$$\bar{K}(X, Y) Z = K(X, Y) Z - A_{B(Y,Z)}(X) + A_{B(X,Z)}(Y)$$
$$+ (\tilde{\nabla}_X B)(Y, Z) - (\tilde{\nabla}_Y B)(X, Z).$$

Lemma 2.1—If M is an invariant submanifold of a conforml K-contact Riemannian manifold $\bar{M}$, then its second fundamental form B satisfies $B(X, \xi) = 0$ for any $X \in X(M)$.

PROOF : Since $\bar{\xi}$ is tangent to M everywhere on M, we have

$$\bar{\nabla}_X \bar{\xi} = \bar{\nabla}_X \xi = \nabla_X \xi + B(X, \xi). \quad \ldots(2.3)$$

Since by (1.6), $\bar{\Delta}_X \xi$ is tangent to M for any $X \in X(M)$, then by taking the normal parts of 2.3), we get $B(X, \xi) = 0$.

Lemma 2.2 —Any invariant submanifold M with induced structure tensors of a conformal K-contact Riemannian manifold $\bar{M}$ is also conformal K-contact.

PROOF : From (2.2) and Lemma 2.1, we have

$$\bar{K}(X, \xi)\xi = K(X, \xi)\xi + (\tilde{\nabla}_X B)(\xi, \xi) - (\tilde{\nabla}_\xi B)(X, \xi). \quad \ldots(2.4)$$

Again from (2.1) and Lemma 2.1, we get

$$(\tilde{\nabla}_X B)(\xi, \xi) = 0 \quad \ldots(2.5)$$

and

$$(\tilde{\nabla}_\xi B)(X, \xi) = 0. \quad \ldots(26)$$

Finally using (2.5) (2.6) and (1.7) in (2.4) we obtain

$$K(X, \xi)\xi = \bar{K}(X, \xi)\xi = X - \eta(X)\xi + (\xi\alpha)\eta(X)\xi - (X\alpha)\xi.$$

This shows that M is a conformal K-contact Riemannian manifold.

Lemma 2.3—Let M be an invariant submanifold of a conformal K-contact Riemannian manifold $\bar{M}$. Then $\bar{K}(X, \xi) Y$ is tangent to M if and only if $\bar{\phi} B(X, Y) = B(X, \phi Y)$ for any $(X, Y \in X(M)$.

PROOF : It is seen in Lemma 2.2 that M is also conformal K-contact Riemannian manifold. Since ξ is a conformal killing vector field on M and $\bar{M}$, by virtue of (1.7) we have

$$\bar{K}(X, \bar{\xi}) Y = (\bar{\nabla}_X \bar{\phi}) Y + \bar{g}(X, Y)(\bar{\xi} \alpha) \bar{\xi} - (X \alpha) Y \quad \ldots(2.7)$$

and

$$K(X, \xi) Y = (\nabla_X \phi) Y + g(X, Y)(\xi \alpha) \xi - (X \alpha) Y. \quad \ldots(2.8)$$

On the otherhand, from (1.9) we have

$$\bar{\nabla}_X (\bar{\phi} Y) = \nabla_X (\phi Y) + B(X, \phi Y) = (\nabla_X \phi) Y + \phi(\nabla_X Y) + B(X, \phi Y) \quad \ldots(2.9)$$

and

$$\bar{\nabla}_X (\bar{\phi} Y) = (\bar{\nabla}_X \bar{\phi}) Y + \bar{\phi}(\bar{\nabla}_X Y) = (\bar{\nabla}_X \bar{\phi}) Y + \bar{\phi}(\nabla_X Y + B(X, Y))$$

$$= (\bar{\nabla}_X \bar{\phi}) Y + \phi(\nabla_X Y) + \bar{\phi} B(X, Y). \quad \ldots(2.10)$$

From (2.7), (2.8), (2.9) and (2,10) we have

$$K(X, \xi) Y - g(X, Y)(\xi \alpha) \xi - (X\alpha) Y - \bar{K}(X, \xi) Y$$

$$+ \bar{g}(X, Y)(\bar{\xi} \alpha) \bar{\xi} + (X\alpha) Y = \bar{\phi} B(X, Y) - B(X, \phi Y)$$

which implies

$$K(X, \xi) Y - \bar{K}(X, \xi) Y = \bar{\phi} B(X, Y) - B(X, \phi Y)$$

Hence the Lemma. Q.E.D.

Lemma 2.4—For an invariant submanifold M of a conformal K-contact Riemannian manifold $\bar{M}$, if the vector field X on M is orthogonal to $\bar{\xi}$, then we have

$$\bar{\phi} \bar{K}(\bar{\xi}, N) X = -\bar{K}(\xi, N) \bar{\phi} X - 2(N\alpha) \bar{\phi} X$$

where N is a vector field normal to M.

PROOF : From (1.1), we find

$$\bar{\nabla}_N (\bar{\phi}^2 X) = -\bar{\nabla}_N X + \bar{g}(\bar{\nabla}_N X, \bar{\xi}) \bar{\xi} + \bar{g}(X, \bar{\nabla}_N \bar{\xi}) \bar{\xi}$$

$$+ \bar{g}(X, \bar{\xi}) \bar{\nabla}_N \bar{\xi}.$$

On the other hand, we have

$$\bar{\nabla}_N (\bar{\phi}^2 X) = (\bar{\nabla}_N \bar{\phi}) \bar{\phi} X + \bar{\phi}(\bar{\nabla}_N \bar{\phi}) X + \bar{\phi}^2 \bar{\nabla}_N X$$

$$= \bar{K}(N, \bar{\xi}) \bar{\phi} X - (\bar{\xi} \alpha) g(N, \bar{\phi} X) + (N\alpha) \bar{\phi} X$$

$$+ \bar{\phi} \bar{K}(N, \bar{\xi}) X - (\bar{\xi} \alpha) \bar{g}(N, X) \bar{\phi} \bar{\xi} + (N\alpha) \bar{\phi} X$$

$$- \bar{\nabla}_N X + \bar{g}(\bar{\nabla}_N X, \bar{\xi}) \bar{\xi}.$$

Thus we have,

$$\bar{g}(X, \bar{\nabla}_N \bar{\xi})\, \bar{\xi} + \bar{g}(X, \bar{\xi})\, \bar{\nabla}_N \bar{\xi} = \bar{K}(N, \bar{\xi})\, \bar{\phi} X - (\bar{\xi}\alpha)\, \bar{g}(N, \bar{\phi} X)\, \bar{\xi}$$

$$+ 2(N\alpha)\, \bar{\phi} X + \bar{\phi}\, \bar{K}(N, \bar{\xi})\, X$$

from which

$$0 = \bar{\eta}(X)\, (\bar{\phi} N + \alpha N) + \bar{g}(X, \bar{\phi} N + \alpha N)\, \bar{\xi}$$

$$= \bar{K}(N, \bar{\xi})\, \bar{\phi} X + \bar{\phi}\, \bar{K}(N, \bar{\xi})\, X + (\bar{\xi}\alpha)\, \bar{g}(N, \bar{\phi} X)\, \bar{\xi} + 2(N\alpha)\, \bar{\phi} X$$

$$= \bar{K}(N, \bar{\xi})\, \bar{\phi} X + \bar{\phi}\, \bar{K}(N, \bar{\xi})\, X + 2(N\alpha)\, \bar{\phi} X$$

Theorem 2.1—Any invariant submanifold M of a conformal K-contact Riemannian manifold $\bar{M}$ is minimal.

PROOF : From the proof of Lemma 2.3, we have

$$\bar{g}(\bar{K}(X, \xi)\, Y, N) = \bar{g}(B(X, \phi Y), N) - \bar{g}(\bar{\phi}\, B(X, Y), N)$$

$$= \bar{g}(A_N X, \phi Y) + \bar{g}(A_{\bar{\phi} N} X, Y)$$

where A_N is defined to be $g(B(X, Y), N) = g(A_N X, Y)$.

Replacing Y by ϕY, we find

$$\bar{g}(A_N X, Y) = -\bar{g}(\bar{K}(X, \xi)\, \phi Y, N) + g(A_{\bar{\phi} N} X, \phi Y).$$

Taking a ϕ-basis $\{\xi; e_1, \ldots, e_m : \phi e_1, \ldots, \phi e_m\}$
we have

$$\operatorname{Tr} A_N = -\sum_{i=1}^{m} \bar{g}(\bar{K}(e_i, \xi)\, \phi e_i, N)$$

$$-\sum_{i=1}^{m} \bar{g}\, \bar{K}(\phi e_i, \xi)\, \phi^2 e_i, N) - \operatorname{Tr} A_N \phi$$

$$= \sum_{i=1}^{m} \bar{g}(\bar{K}(\phi e_i, e_i)\, \xi, N)$$

since ϕ is skew-symmetric, $\operatorname{Tr} A_N \phi$ vanishes identically and by virtue of Bianchi's identity.

On the other hand, by Lemma 2.4, we get

$$\bar{g}(\bar{K}(\phi e_i, e_i)\, \xi, N) = \bar{g}(\bar{K}(\xi, N)\, \phi e_i, e_i)$$

$$= -\bar{g}(\bar{\phi}\, K(\xi, N)\, e_i, e_i) - 2(N\alpha)\, g(\phi e_i, e_i)$$

(*equation continued on p.* 1124)

$$= - \bar{g} (\bar{\phi} K (\xi, N) e_i, e_i)$$

$$= \bar{g} (\bar{K} (\xi, N) e_i, \phi e_i)$$

$$= \bar{g} (\bar{K} (e_i, \phi e_i), \xi, N)$$

$$= - \bar{g} (\bar{K} (e_i, e_i) \xi, N)$$

from which $\bar{g} (\bar{K} (\phi e_i, e_i) \xi, N) = 0$. Thus we have Tr $A_N = 0$ for all N, i. e. Tr $B = 0$, which shows that M is minimal

We consider the second fundamental form B of M as a normal bundle valued symmetric 2-form

$$B : T_x (M) \times T_x (M) \to T_x (M)^{\perp} \text{ at each } x \in M.$$

Then for any $X, Y, Z, W \in X (M)$, we have

$$(\tilde{K} (X, Y) \circ B) (Z, W) = K^{\perp} (X, Y) (B (Z, W)) - B (K (X, Y) Z, W) - B (Z, K (X, Y) W)$$

Putting $Y = Z = \xi$, and using Lemma 2.1 and (1.8), we get

$$(K (X, \xi). B) (\xi, W) = K^{\perp} (X, \xi) (B (\xi, W)) - B (K (X, \xi) (\xi, W) - B (\xi, K (X, \xi) W) = - B (X, W).$$

Then it follows that

Proposition 2.1—Let M be an invariant submanifold of a conformal K-contact Riemannian manifold $\bar{M}$. Then M is minimal if any only if

$$\sum_{i=1}^{2m+1} (\tilde{K} (V_i, \xi). B) (\xi, V_i) = 0, \text{ where } (V_1, \ldots, V_{2m+1}) \text{ is a frame in } T_x(M).$$

Proposition 2.2—Let M be an invariant submanifold of a conformal K-contact Riemannian manifold $\bar{M}$. Then M is totally geodesicif and only if $\tilde{K} (X, \xi). B = 0$ for any $X \in X (M)$.

From Propositions 2.1 and 2.2, we have

Proposition 2.3—Let M be an invariant submanifold of a conformal K-contact Riemannian manifold $\bar{M}$. Then M is totally geodesic if and only if its second fundamental form B is covariantly constant.

ACKNOWLEDGEMENT

Authors are indebted to referee for valuable suggestions in improving this paper.

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