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ON THE GENERATING FUNCTIONS AND PARTIAL SUMS OF THE FOURIER SERIES

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The author has obtained some asymptotic expressions for partial sums of a Fourier series. The results show that certain conditions as imposed on the generating functions of the Fourier series in an earlier paper for a Riesz summability and for the convergence of the series are indeed not only sufficient but necessary as well.

1. INTRODUCTION

Let L denote the space of all 2π-periodic functions which are Lebesgue-integrable over $[0, 2\pi]$ and let $s_n(f, x)$ be the nth partial sum of the Fourier series of $f \in L$ at a point x. For real numbers x, s and d we write

$$\phi(t) = \tfrac{1}{2}\{f(x+t) + f(x-t) - 2s\} \qquad ...(1.1)$$

$$\phi_1(t) = (1/t)\int_0^t \phi(u)\, du \qquad ...(1.2)$$

$$P(t) = \phi(t) - \phi_1(t) \qquad ...(1.3)$$

and for a specified function F,

$$\mathscr{P}_d(F) = \{f \in L/\phi_1 \in L \text{ and } \lim_{t\to 0+} t^{-1} F(1/t) \int_0^t |P(u)|\, du = d\}. \qquad ...(1.4)$$

In an attempt to improve the well-known Hardy-Littlewood criterion (see Hardy and Littlewood[6]) for the convergence of Fourier series at a point x, we[4] proved the following :

Theorem A—Let $\phi_1(t) = o(1)\ (t \to 0+)$. .. (1.5)

Then in order that $(s_n(f; x)) \in (R, \exp(\omega^\alpha), \beta)$ to s, where $0 < \alpha < 1$, $\beta > 0$ and $f \in \mathscr{P}_d(\log)$, it is sufficient that

$$f \in \mathscr{P}_0(\log) \qquad ...(1.6)$$

In the same paper, we[4] have also proved the following theorem which improves an earlier result due to Mohapatra[8] :

Theorem B—Let (1.5) hold. Then in order that $(s_n(f; x)) \in (R, \exp\{\log \omega)^{\Delta}\}, \beta)$ to s, where $f \in \mathscr{P}_d$ (log log), $\beta > 0$ and $\Delta > 0$ however large, it is sufficient that

$$f \in \mathscr{P}_0 \text{ (log log)}. \qquad ...(1.7)$$

Wang[9] proved that the condition (1.5) alone was not sufficient to ensure summability of $(s_n(f; x))$ by either method. In fact he[9] gave an example of an even function satisfying

$$\phi_1(t) = o\left\{\left(\log \frac{1}{t}\right)^{-1}\right\} \quad (t \to 0+) \qquad ...(1.8)$$

whose Fourier series at $x = 0$ diverges. On the other hand the condition (1.8) is not necessary which follows fícm Izuml *et al.*[7] (see also Bary[1], p. 285). Now the question arises as to whether or not the conditions (1.6) and (1.7) are necessary whenever $f \in \mathscr{P}_d(F)$ satisfies (1.5). In this note we answer this question in affirmative. In fact we prove a general result from which we deduce that if, for $f \in \mathscr{P}_d$ (log),

$$\lim_{t \to 0+} P(t) \log(1/t) = d \neq 0$$

then $s_n(f; x) \sim -d \log \log n \; (n \to \infty)$, that is the Fourier series of f at x diverges to ∞ for $d < 0$ and hence it cannot be summable by any regular summability method. Consequently the condition

$$P(t) = o(1/\log(1/t)) \; (t \to 0+)$$

and hence (1.6) is necessary whenever $f \in \mathscr{P}_d$ (log) satisfies (1.5). Similarly, it follows that (1.7) is necessary for $f \in \mathscr{P}_d$ (log log) satisfying (1.5).

We now state the results which we intend to prove.

Theorem 1—Suppose F is a positive and increasing function defined on $(0, \infty)$ and is such that

$$\alpha_n = \int_{1/\pi}^{n} (t\,F(t))^{-1}\,dt \to \infty, \text{ as } n \to \infty. \qquad ...(1.9)$$

Then in order that, for $f \in \mathscr{P}_d(F)$

$$s_n(f; x) = o(\alpha_n) \; (n \to \infty) \qquad ...(1.10)$$

it is necessary and sufficient that

$$f \in \mathscr{P}_0(F). \qquad ..(1.11)$$

To prove the necessity part of Theorem 1, we first prove.

Theorem 2—Let F and α_n be the same as defined in Theorem 1 and let

$$\lim_{t \to 0+} P(t)\,F(1/t) = d \neq 0. \qquad ...(1.12)$$

Then

$$s_n(f;x) \sim -d\chi_n \qquad \text{...(1.13)}$$

We shall write

$$K_n(t) = \frac{\sin(n+\frac{1}{2})t}{2\sin\frac{1}{2}t} - \sum_{m=1}^{n} \frac{\sin mt}{mt}$$

and shall also make use of the following :

$$\sum_{m=1}^{n} \frac{\sin mt}{m} = \tfrac{1}{2}(\pi - t) + O\{(nt)^{-1}\} \ (0 < t < 2\pi) \qquad \text{...(1.14)}$$

$$\sum_{m=1}^{\infty} \frac{\sin mt}{m} = \tfrac{1}{2}(\pi - t) \ (0 < t < 2\pi). \qquad \text{...(1.15)}$$

The result (1.15) is well-known (e.g. see Bromwich[2], p. 356) and (1.14) may be obtained from it.

2. Proof of Theorem 1

The necessity part of the theorem follows from Theorem 2 so we prove only the sufficiency part.

Proceeding as in Chandra[5], we get

$$s_n(f;x) - s - \frac{1}{\pi}\int_0^{\pi} \phi_1(t)\,dt = \frac{2}{\pi}\int_0^{\pi} P(t)\,K_n(t)\,dt$$

$$= \frac{2}{\pi} J, \text{ say.}$$

Let $\epsilon > 0$ be given. Then there exists $\delta = \delta(\epsilon) > 0$ such that

$$\int_0^t |P(u)|\,du < \epsilon\, t/F(1/t) \text{ for } 0 < t \leqslant \delta < 1. \qquad \text{...(2.1)}$$

Then for $n > \delta^{-1}$, we write

$$J = \int_0^{1/n} + \int_{1/n}^{\delta} + \int_{\delta}^{\pi} = J_1 + J_2 + J_3, \text{ say.}$$

We now observe that for all $t \in (0, \pi)$

$$|K_n(t)| \leqslant 3n + 1$$

therefore, by (2.1),

$$| J_1 | \leqslant (3n + 1) \int_0^{1/n} | P(u) | \, du < 4 \epsilon / F(n). \qquad \ldots(2.2)$$

Also for all n and uniformly in $t \in (0, \pi)$

$$K_n(t) = O(t^{-1}). \qquad \ldots(2.3)$$

Thus by using (2.3) and the fact that $\delta > 0$ is fixed, we get

$$J_3 = O(1) \int_\delta^\pi t^{-1} | P(t) | \, dt = O(1). \qquad \ldots(2.4)$$

Once again, by using (2.3), we get

$$J_2 = O(1) \int_{1/n}^\delta t^{-1} \; | P(t) | \, dt. \qquad \ldots(2.5)$$

Integrating by parts and using (2.1), we get

$$\int_{1/n}^\delta t^{-1} | P(t) | \, dt < 2\epsilon / F(1/\delta) + \epsilon \int_{1/n}^\delta (tF(1/t))^{-1} \, dt$$

$$= \epsilon \, [2/F(1/\delta) + \alpha_n]$$

by (1.9).

Thus collecting the results and letting $n \to \infty$, we get

$$s_n(f; x) = s + \frac{1}{\pi} \int_0^\pi \phi_1(t) \, dt + o(\alpha_n) = o(\alpha_n)$$

since $\phi_1(t) \in L$ and hence the integral $(1/\pi) \int_0^\pi \phi_1(t) \, dt$ is finite.

This completes the proof of Theorem 1.

3. Proof of Theorem 2

As in Theorem 1,

$$s_n(f; x) = s + \frac{1}{\pi} \int_0^\pi \phi_1(t) \, dt + \frac{2}{\pi} J \qquad \ldots(3.1)$$

where

$$\frac{2}{\pi} J = \frac{2}{\pi} \int_0^\pi \left[P(t) \, F\left(\frac{1}{t}\right) - d \right] \frac{K_n(t)}{F(1/t)} \, dt + \frac{2d}{\pi} \int_0^\pi \frac{K_n(t)}{F(1/t)} \, dt. \qquad \ldots(3.2)$$

Now, since $P(t)\,F(1/t) - d = o(1)$ as $t \to 0+$, therefore proceeding as in Theorem 1, we may obtain that

$$\frac{2}{\pi}\int_0^{\pi} \{P(t)\,F(1/t) - d\}\, \frac{K_n(t)}{F(1/t)}\, dt = o(\alpha_n)\ (n \to \infty). \qquad \ldots(3.3)$$

We have

$$\frac{2d}{\pi}\int_0^{\pi} \frac{K_n(t)}{F(1/t)}\, dt = \frac{2d}{\pi}\int_0^{\pi} \frac{1}{F(1/t)}\, \frac{\sin(n+\frac{1}{2})\,t}{2\sin\frac{1}{2}t}\, dt$$

$$- \frac{2d}{\pi}\int_0^{\pi} \frac{1}{F(1/t)}\left(\sum_{m=1}^{n} \frac{\sin mt}{mt}\right) dt.$$

The first integral on the right and the integral

$$- \frac{2d}{\pi}\int_0^{1/n} \frac{1}{F(1/t)}\left(\sum_{m=1}^{n} \frac{\sin mt}{mt}\right) dt$$

are bounded for all $n \geqslant 1$ and also by (1.14)

$$- \frac{2d}{\pi}\int_{1/n}^{\pi} \frac{1}{F(1/t)}\left(\sum_{m=1}^{n} \frac{\sin mt}{mt}\right) dt$$

$$= -\frac{2d}{\pi}\int_{1/n}^{\pi} (tF(1/t))^{-1}\, \{\tfrac{1}{2}(\pi - t)\}\, dt$$

$$+ O(n^{-1}) \int_{1/n}^{\pi} (t^2\, F(1/t))^{-1}\, dt$$

$$= -d\,\alpha_n + (2d/\pi)\int_{1/n}^{\pi} (F(1/t))^{-1}\, dt + O(1)$$

$$= -d\,\alpha_n + O(1).$$

Therefore, collecting the above results, we get

$$\frac{2d}{\pi}\int_0^{\pi} \frac{K_n(t)}{F(1/t)}\, dt = -d\,\alpha_n + O(1). \qquad \ldots(3.4)$$

Combining (3.1) through (3.4), we get

$$s_n(f;x) = s + (1/\pi)\int_0^{\pi}\phi_1(t)\,dt - d\,\alpha_n + O(1)$$

$$= -\,d\,\alpha_n + o(\alpha_n)$$

as $n \to \infty$.

This completes the proof of Theorem 2.

Remark : We remark that if for some $f \in L$

$$\lim_{t\to 0+} P(t)\,F(1/t) \qquad \ldots(3.5)$$

oscillates between some finite real numbers then Theorem 2 is not applicable. In fact it may happen that Fourier series of such an $f \in L$ may converge. For example, let f be an even function and $x = 0$, $s = 0$. Then $\phi(t) = f(t)$. Suppose, for $k > \pi e^2$,

$$\phi(t) = \frac{\sin\left(\log\log \frac{k}{t}\right)}{\log\log \frac{k}{t}} - \frac{\cos\left(\log\log \frac{k}{t}\right)}{\log \frac{k}{t}\,\log\log \frac{k}{t}} \quad (0 < t \leqslant \pi)$$

$$= \alpha(t) - \beta(t), \text{ say.}$$

Then it is easy to verify that

$$P(t) = \frac{1}{t}\int_0^t \frac{\sin\left(\log\log \frac{k}{u}\right)}{\log \frac{k}{u}\left(\log\log \frac{k}{u}\right)^2}\,du - \beta(t).$$

Hence for $F(1/t) = \log(k/t)\log\log(k/t)$

$$P(t)\,F(1/t) = -\cos\left(\log\log \frac{k}{t}\right)$$

$$+\, t^{-1}F(1/t)\int_0^t \frac{\sin\left(\log\log \frac{k}{u}\right)}{\log \frac{k}{u}\left(\log\log \frac{k}{u}\right)^2}\,du.$$

Thus limit in (3.5) oscillates between 1 and -1 but, as it follows from Chandra[3], the Fourier series of this f converges to zero.

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