Indian J. pure appl. Math., **20** (11) : 1146–1150, November 1989

UNSTEADY MOTION OF A SEMI-INFINITE CONDUCTING LIQUID BY A SUDDENLY APPLIED VELOCITY ON ITS SURFACE

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(*Received* 5 *December* 1988; *accepted* 17 *May* 1989)

The motion set up in a semi-infinite incompressible viscous conducting liquid by a suddenly applied velocity over a circular area on the free surface of the liquid in presence of magnetic fields acting radially and axially has been obtained. Exact solutions are obtained for both the cases and are presented in the forms of infinite series and infinite integrals.

1. INTRODUCTION

Edward[1] investigated the duct flow of a conducting fluid under circular and radial magnetic fields. Globe[2] solved the problem for a complete annulus under a radial field. Such a field can be produced in a liquid by a line source. Goble[2] and Elco *et al.*[3] pointed out how such a field can be generated in practice. Sengupta and Mahapatra[4] considered the problem of a semi-infinite medium of conducting liquid set in motion by an impulsive velocity on its surface.

In the present paper it is proposed to consider the unsteady rotational motion set up in a semi-infinite medium of viscous incompressible conducting liquid by a suddenly applied velocity within a circular area of the free surface in presence of a radial and axial magnetic fields. In fact, the liquid is contained in the annular region between two circular co-axial cylinders, the inner cylinder having an infinitesimal small circular section and the outer cylinder having a very large circular cross-section. The axis of Z is taken along the common axis of the cylinders and it points into the medium. Solutions are obtained in the forms of infinite series and infinite integrals. Numerical results for the velocity are shown in graphical forms.

2. BASIC EQUATIONS AND BOUNDARY CONDITIONS

Problem 1 : The Medium is under a Radial magnetic Field

Let (r, θ, z) be the cylindrical coordinates of a point in the liquid having the origin on its surface. Let us suppose that the medium is under the action of a radial magnetic field H_0/r. If (u, v, w) be the velocity components in (r, θ, z) directions, then for rotationally symmetric motion of the liquid $u = w = 0$ and $v = v(r, z, t)$. The linearized equation of motion, in this case, is

$$\frac{\partial^2 v}{\partial r^2}+\frac{1}{r}\frac{\partial v}{\partial r}+\frac{\partial^2 v}{\partial z^2}-\frac{n^2}{r^2}v=\frac{1}{c^2}\frac{\partial v}{\partial t} \qquad \text{...(2.1)}$$

where

$$c^2=\frac{\mu}{\rho},\ n^2=1+\frac{\sigma\mu_e^2\ H_0^2}{\mu}.$$

μ is the coefficient of viscosity, ρ the density, σ the conductivity, μ_e the mag netic permeability and H_0 is constant.

Problem 2 : The Medium is under an Axial Magnetic Field

Here we suppose that the liquid is acted on by an axial magnetic field H_0 instead of a radial one. In this case the linearised equation of motion is

$$\frac{\partial^2 v}{\partial r^2}+\frac{1}{r}\frac{\partial v}{\partial r}+\frac{\partial^2 v}{\partial z^2}-\left(q^2+\frac{1}{r^2}\right)v=\frac{1}{c^2}\frac{\partial v}{\partial t} \qquad \text{...(2.2)}$$

where

$$q^2=\frac{\sigma\mu_e^2\ H_0^2}{\mu}.$$

Boundary conditions—We suppose that the rotational motion of the conducting liquid is set up by prescribing suddenly applied velocity within a circu lar area of the free surface. Thus the boundary conditions for both the problems are

$$v=0 \text{ as } z\to\infty$$

$$v=f(r)\,H(t) \text{ when } z=0 \qquad \text{...(2.3)}$$

where

$$f(r)=\epsilon r \text{ when } r\leqslant a$$
$$=0 \text{ when } r>a \qquad \text{...(2.4)}$$

and $H(t)$ is the Heavislde unit function.

3. Solutions of the Problem

Problem 1—Let us suppose that a solution of the differential equation (2.1) can be taken in the form

$$\phi(\xi, z, t)=\int_0^\infty v(r, z, t)\,J_n(r\xi)\,dr.$$

Then the use of this transform to eqn. (2.1) gives

$$\left(\frac{d^2}{dz^2}-\xi^2\right)\phi=\frac{1}{c^2}\frac{\partial\phi}{\partial t} \qquad \text{...(3.1)}$$

provided that $r\,v \to 0$ as $r \to 0$ and ∞.

Also if we assume that a bar over a function denote its Laplace transform with p as parameter then the equation (3.1) leads to

$$\left(\frac{d^2}{dz^2} - \xi^2 - \frac{p}{c^2}\right)\bar{\phi}\,(\xi, z, p) = 0 \qquad \ldots(3.2)$$

provided $\phi = 0$ at $t = 0$.

Solution of the equation (3.2) finite for $r \to \infty$ is

$$\bar{\phi}\,(\xi, z, p) = A\,(\xi, p)\exp\left(-\frac{z}{c}\sqrt{c^2\,\xi^2 + p}\right). \qquad \ldots(3.3)$$

Now, we express the function $f(r)$ given in (2.4) as a Fourier-Bessel integral in the form

$$f(r) = \int_0^\infty \xi\, J_n\,(r\xi)\,\left[\int_0^\infty y\,f(y)\,J_n\,(y\theta)\,dy\right]d\xi$$

and replacing $f(y)$ by its actual form given in (2.4) and then converting the result to a series we obtain

$$f(r) = 2\epsilon a^2 \int_0^\infty J_n\,(r\xi) \sum_{m=0}^{\infty} \frac{(-1)^m\,(\xi a/2)^{n+2m+1}}{m!\,(n+2m+3)\,\Gamma\,(n+2m+1)}\,dr. \ldots(3.4)$$

Hence, with the aid of (3.3) and (3.4) and the boundary conditon (2.3) we get

$$A\,(\xi, p) = \frac{2\epsilon\,a^2}{p} \sum_{m=0}^{\infty} \frac{(-1)^m\,(\xi a/2)^{n+2m+1}}{m!\,(n+2m+3)\,\Gamma\,(n+2m+1)}. \qquad \ldots(3.5)$$

Substituting this value of A in (3.3) and using Laplace inversion theorem we get the expression for $\phi\,(\xi, z, t)$ which leads to

$$v\,(r, z, t) = \epsilon\,a^2 \sum_{m=0}^{\infty} \frac{(-1)^m\,a^{n+2m+1}}{2^{n+2m+1}\,m!\,(n+2m+3)\,\Gamma\,(n+2m+1)}$$

$$\times \int_0^\infty \xi^{n+2m+1}\left[e^{-\xi z}\,\mathrm{erfc}\left(\frac{z}{2c\sqrt{t}} - c\,\xi\sqrt{t}\right)\right.$$

$$\left. + e^{\xi z}\,\mathrm{erfc}\left(\frac{z}{2c\sqrt{t}} + c\xi\sqrt{t}\right)\right] J_n\,(\xi\,r)\,d\xi. \qquad \ldots(3.6)$$

Problem 2—Here we assume a solution of the differential equation (2.2) in the form

$$v = \int_0^\infty \psi\,(\xi, z, t)\,J_1\,(\xi r)\,d\xi \qquad \ldots(3.7)$$

where ψ satisfies the differential equation

$$\frac{\partial^2 \psi}{\partial z^2} = (q^2 + \xi^2)\,\psi + \frac{1}{c^2}\frac{\partial \psi}{\partial t}. \qquad \ldots(3.8)$$

Applying Laplace transform to eqn. (3.8) subject to $\psi = 0$ at $t = 0$ and the boundary condition (2.3) we get after necessary calculations

$$\bar{\psi}(\xi, z, p) = \frac{\epsilon\, a^2}{p}\, J_2(\xi a) \exp\left(- \sqrt{c^2 \eta^2 + p}\, \right) \qquad \ldots(3.9)$$

where $\eta^2 = q^2 + \xi^2$. Laplace inversion of (3.9), with the help of (3.7) gives

$$v(r, z, t) = \tfrac{1}{2}\, \epsilon\, a^2 \int_0^\infty \left[e^{-\eta z} \operatorname{erfc}\left(\frac{z}{2c\sqrt{t}} - c\eta\sqrt{t} \right) + e^{\eta z} \operatorname{erfc}\left(\frac{z}{2c\sqrt{t}} + c\,\eta\sqrt{t} \right) \right] J_2(\xi a)\, J_1(\xi r)\, d\xi. \qquad \ldots(3.10)$$

4. Numerical Results

It is clear from (3.6) and (3.10) that in either case, the initial response of the velocity is infinitely small and it increases with time giving a steady response after an infinite interval of time. Taking mercury as the relevant conducting fluid, the nature of the velocity has been shown in Fig. 1 (for problem 1) and in Fig. 2 (for problem 2). For numerical calculations we have chosen $a = 1$, $z = 1$, $r = 0.5$, $c = 0.33215$ and $n = 1, 2, 3$ $n = 1$ gives the nature for non-MHD case while $n = 2{,}3$ show the distribution for MHD case. It is seen that the magnetic field decreases the velocity for both the problems.

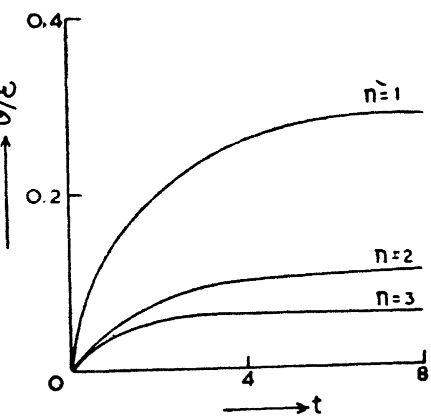

Fig. 1.

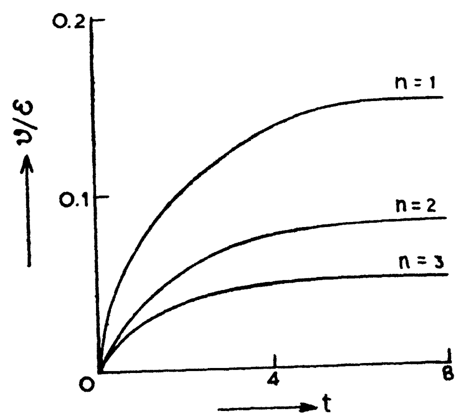

Fig. 2.

It may be noted in this connection that the integrals involved in (3.6) and (3.10) seem to be difficult to evaluate analytically and therefore they are obtained numerically by Filon's method[6].

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