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ON HYDROMAGNETIC TURBULENT SHEAR FLOW

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Hydromagnetic turbulent shear flow of viscous incompressible, electrically conducting fluid between two infinite porous horizontal planes in presence of a uniform transverse magnetic field has been studied by the semi-empirical method. The expressions for the mean distributions for velocity and magnetic field have been obtained when the surfaces of the channel are non-conducting and conducting. Numerical results are shown graphically.

1. INTRODUCTION

It is known that a turbulent state eventually results from the instability of laminar flow in a celestial body like the sun, earth etc., and there are many ways of producing turbulence, for example by thermal instability or by a flow of air through a wind tunnel[1]. Turbulent shear flow of an incompressible viscous fluid between two parallel planes and through a circular pipe has been studied by Pai[2,3] by the semi-empirical approach suggested by Kampe de Ferriet[4]. These theoretical results agree with the experimental results of Laufer[5] and Nikurdse[6]. Jain's[7] solutions for hydromagnetic turbulent shear flow between two parallel non-permeable planes are also in close conformity with the experimental results of Murgatroyd[8]. Mehta and Balasubramanyam[9] also solved a similar type of problem.

In the present paper, it is proposed to study the hydromagnetic turbulent shear flow of an incompressible viscous electrically conducting fluid between two horizontal parallel permeable planes in presence of a uniform transverse magnetic field by the semi-empirical method of Kampe de Ferriet[4]. Two cases are considered : (i) the surfaces of the channel are non-conducting and (ii) the surfaces are conducting of the same conductivity. Due to non-availability of the relevant experimental data, assumptions are made regarding the numerical values of the constants. The expressions for the velocity and the magnetic field are obtained and their natures are shown in graphical forms for both the cases.

2. GOVERNING EQUATIONS

We consider the fully developed steady state hydromagnetic turbulent shear flow of an incompressible viscous electrically conducting fluid between two uniformly porous parallel planes at a distance $2L$ apart. Let the x-axis be in the direction of the flow parallel to the planes, the y-axis normal to the planes and the z-axis transverse to both x and y. Let the middle plane be $y = 0$ and the hydromagnetic flow variables are functions of y only. The planes of the channel are now $y = \pm L$.

Neglecting displacement currents, the hydromagnetic equations in e.m. units are[9]

$$v_j \frac{\partial v_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 v_i}{\partial x_j \partial x_j} + h_j \frac{\partial h_i}{\partial x_j} - \frac{1}{2}\frac{\partial h_j^2}{\partial x_i} \quad ...(2.1)$$

$$v_j \frac{\partial h_i}{\partial x_j} - h_j \frac{\partial v_i}{\partial x_j} = \nu_H \frac{\partial^2 h_i}{\partial x_j \partial x_j} \quad ...(2.2)$$

$$\frac{\partial v_i}{\partial x_i} = 0 \quad ...(2.3)$$

$$\frac{\partial h_i}{\partial x_i} = 0 \quad ...(2.4)$$

where $(i, j = 1, 2, 3)$, $(x_1, x_2, x_3) = (x, y, z)$, $h_i = H_i/\sqrt{4\pi\rho}$, H_i are the magnetic field intensity components, v_i velocity components, p pressure, ρ density, ν kinetic viscosity, ν_H $(= 1/4\,\pi\sigma)$ magnetic diffusivity and σ is the electrical conductivity.

Let the flow be composed of a mean motion with superimposed random fluctuations and eqns. (2.1) to (2.4) are satisfied by the instantaneous flow variables, which may be expressed as

$$f = \bar{f} + f' \quad ...(2.5)$$

where $\bar{f}$ and f' denote the mean and flutuating parts of the flow variable respectively.

Substituting (2.5) into equs. (2.1) to (2.4) we get[9]

$$\bar{v}_j \frac{\partial \bar{v}_i}{\partial x_j} - \bar{h}_j \frac{\partial \bar{h}_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x_i} + \nu \frac{\partial^2 \bar{v}_i}{\partial x_j \partial x_j} - \frac{\partial b_{ij}}{\partial x_j} - \frac{1}{2}\left(\frac{\partial \bar{h}_j^2}{\partial x_i} + \frac{\partial h_j'^2}{\partial x_i}\right) \quad ...(2.6)$$

$$\bar{v}_j \frac{\partial \bar{h}_i}{\partial x_j} - \bar{h}_j \frac{\partial \bar{v}_i}{\partial x_j} = \nu_H \frac{\partial^2 \bar{h}_i}{\partial x_j \partial x_j} + \frac{\partial a_{ij}}{\partial x_j} \quad ...(2.7)$$

$$\frac{\partial \bar{v}_i}{\partial x_i} = 0 \quad ...(2.8)$$

$$\frac{\partial \bar{h}_i}{\partial x_i} = 0 \quad ...(2.9)$$

where

$$\left.\begin{aligned} a_{ij} &= \langle v'_i h'_j \rangle - \langle v'_j h'_i \rangle \\ b_{ij} &= \langle v'_i v'_j \rangle - \langle h'_i h'_j \rangle \end{aligned}\right\} \quad ...(2.10)$$

and $< >$ denote average value.

Let us take $\bar{v}_i = \{\bar{v}_x(y), \bar{v}_y(y), 0\}$, $\bar{h}_i = \{\bar{h}_x(y), \bar{h}_y(y), 0\}$, the components of a_{ij} and b_{ij} are functions of y only and assume that there is a uniform transverse magnetic field h_0 perpendicular to the main flow direction. Then equations (2.8) and (2.9) give

$$\bar{v}_y = \text{constant} = v_0, \; \bar{h}_y = \text{constant} = h_0. \quad ...(2.11)$$

Introducing the non-dimensional quantities

$$\xi = \frac{x}{L}, \eta = \frac{y}{L}, V = \frac{\bar{v}_x}{u^*}, H = \frac{\bar{h}_x}{u^*}$$

$$A_{ij} = \frac{a_{ij}}{u^{*2}}, B_{ij} = \frac{b_{ij}}{u^{*2}}, R^* = \frac{Lu^*}{\nu}, R^*_M = \frac{Lu^*}{\nu_H} \quad ...(2.12)$$

$$R = \frac{Lv_0}{\nu}, \text{ cross-flow Reynold number}$$

$$R_M = \frac{Lh_0}{\nu_H} \text{ Magnetic Reynold number}$$

$$\epsilon = \frac{\nu}{\nu_H}, \tilde{\omega} = \frac{\bar{p} - \bar{p}_0}{\rho u^{*2}}, \chi = \frac{1}{u^{*2}} \left\langle \left(h'^2_x + h'^2_y + h'^2_z \right) \right\rangle$$

where $u^* = \sqrt{\tau/\rho}$ is the reference velocity, τ is the shearing stress on the plane $y = L$ (i.e. on $\eta = 1$) and $\bar{p}_0$ is the reference pressure which is taken as the mean pressure at $\xi = 0$, $\eta = 1$, the hydromagnetic equations (2.6) and (2.7) reduce to

$$\frac{d^2V}{d\eta^2} - R\frac{dV}{d\eta} + \frac{R_M}{\epsilon}\frac{dH}{d\eta} = R^*\left(\frac{\partial\tilde{\omega}}{\partial\xi} + \frac{dB_{xy}}{d\eta}\right) \quad ...(2.13)$$

$$\frac{\partial\tilde{\omega}}{\partial\eta} + \frac{d}{d\eta}\left[B_{yy} + \frac{1}{2}(H^2 + \chi)\right] = 0 \quad ...(2.14)$$

$$\frac{dB_{yz}}{d\eta} = 0, \; \frac{dB_{zx}}{d\eta} = 0 \quad ...(2.15)$$

$$\frac{d^2H}{d\eta^2} - \epsilon R\frac{dH}{d\eta} + R_M\frac{dV}{d\eta} + R^*_M\frac{dA_{xy}}{d\eta} = 0. \quad ...(2.16)$$

3. Solutions for the Problem

Case I : The Surfaces are Non-conducting

Here the boundary conditions for the problem are

$$V = H = 0 \quad \text{at} \quad \eta = \pm 1 \qquad \text{...(3.1)}$$

$$A_{ij} = B_{ij} = 0 \quad \text{at} \quad \eta = \pm 1 \qquad \text{...(3.2)}$$

$$\tilde{\omega}(\xi, \eta) = 0 \quad \text{at} \quad \xi = 0, \eta = 1. \qquad \text{...(3.3)}$$

Equations (2.14) and (2.15) when integrated subject to the boundary conditions (3.2) and (3.3) lead to

$$A_{yz} = B_{yz} = 0$$

$$\tilde{\omega}(\xi, \eta) + B_{yy} + \tfrac{1}{2}(H^2 + \chi) = A_0 \xi \qquad \text{.. (3.4)}$$

where $A_0 = \frac{\partial \tilde{\omega}}{\partial \xi}$ is the axial pressure gradient assumed to be given for the flow. From these results we made the same conclusions as in Mehta and Balasubramanyam[9].

In the absence of turbulence $A_{ij} = B_{ij} = 0$ and the solutions of the equations (2.13) and (2.16) satisfying the boundary conditions (3.1) and (3.13) are given by

$$V_l = V_c\,[\epsilon R a_0 + (\epsilon R - \alpha)\, a_1\, e^{\alpha\eta} + (\epsilon R - \beta)\, a_2\, e^{\beta\eta} + a_3 - \epsilon R\eta] \qquad \text{...(3.5)}$$

and

$$H_l = V_c\, R_M\, [a_0 + a_1\, e^{\alpha\eta} + a_2\, e^{\beta\eta}] \qquad \text{...(3.6)}$$

where

$$\alpha, \beta = \tfrac{1}{2}\,[R(1 + \epsilon) \pm \sqrt{\{R^2(1 - \epsilon)^2 + 4M^2\}}]$$

$$a_0 = \frac{\beta \coth \alpha - \alpha \coth \beta}{\alpha - \beta}$$

$$a_1 = -\frac{\beta \operatorname{cosech} \alpha}{\alpha - \beta}$$

$$a_2 = \frac{\alpha \operatorname{cosech} \beta}{\alpha - \beta}$$

$$a_8 = \frac{\alpha\beta(\coth \beta - \coth \alpha)}{\alpha - \beta} \qquad \text{...(3.7)}$$

$V_c = A_0 R^*/\alpha\beta$ is the characteristic velocity and $M = ah_0/\sqrt{\nu\nu_H}$ is the Hartmann number.

In the presence of turbulence, $A_{ij}, B_{ij} \neq 0$ and we may get the solution for the mean velocity distribution V_t for the turbulent shear flow compatiable with the corresponding laminar flow with the same characteristic velocity V_c, by assuming V_t of the form

$$V_t = V_c\,[\epsilon R a_0 + (\epsilon R - \alpha)\, a_1\, e^{\alpha\eta} + (\epsilon R - \beta)\, a_2\, e^{\beta\eta} + a_3 - \epsilon R\eta + A_1\,(\eta + 1)^2 + A_2\,(\eta + 1)^m],\quad m > 2 \qquad ...(3.8)$$

satisfying the boundary condition $V_t = 0$ at $\eta = -1$.

Introducing the empirical parameter

$$s = \frac{\tau_t}{\tau_l} = \frac{\left(\dfrac{dV_t}{d\eta}\right)_{\eta=1}}{\left(\dfrac{dV_l}{d\eta}\right)_{\eta=1}} \qquad ..(3.9)$$

and using the boundary condition $V_t = 0$ at $\eta = 1$ we get

$$A_1 = -\frac{s-1}{2\,(m-2)\,(\alpha-\beta)}\Big[(\alpha-\beta)\,(\alpha\beta - \epsilon R) + \epsilon R a_3 + \alpha\beta\,(\alpha \coth\alpha - \beta\coth\beta)\Big]$$

$$A_2 = \frac{s-1}{2^{m-1}\,(m-2)\,(\alpha-\beta)}\Big[(\alpha-\beta)\,(\alpha\beta - \epsilon R) + \epsilon R a_3 + \alpha\beta\,(\alpha \coth\alpha - \beta\coth\beta)\Big]. \qquad ...(310)$$

For turbulent flow $s > 1$ and for laminar flow $s = 1$. The parameters s and m are to be determined experimentally.

Similarly, we assume the magnetic field for the turbulent flow in the form

$$H_t = V_c\, R_M\,[a_0 + a_1\, e^{\alpha\eta} + a_2\, e^{\beta\eta} - \eta + A_3\,(\eta + 1)^2 + A_4\,(\eta + 1)^m],\quad m > 2 \qquad ...(3.11)$$

which satisfies the boundary condition at $\eta = -1$. Introducing the emperical parameter

$$l = \frac{\left(\dfrac{dH_t}{d\eta}\right)_{\eta=1}}{\left(\dfrac{dH_l}{d\eta}\right)_{\eta=1}} \qquad ...(3.12)$$

and using the boundary condition at $\eta = 1$, we get

$$A_3 = -\frac{(l-1)\,(a_3 - \alpha + \beta)}{2\,(m-2)\,(\alpha-\beta)}$$

$$A_4 = \frac{(l-1)\,(a_3 - \alpha + \beta)}{2^{m-1}\,(m-2)\,(\alpha-\beta)}\,. \qquad ...(3.13)$$

For turbulent flow $l > 1$ and for laminar flow $l = 1$. The parameters l and m are to be determined experimentally.

Case II : The Surfaces are Conducting

We assume that the surfaces $\eta = \pm 1$ of the channel are at rest and they are conducting. Then the boundary conditions for the problem are

$$V = 0 \quad \text{at} \quad \eta = \pm 1$$

$$\phi \frac{dH}{d\eta} \pm H = 0 \quad \text{at} \quad \eta = \pm 1$$

$$A_{ij} = B_{ij} = 0 \quad \text{at} \quad \eta = \pm 1$$

$$\tilde{\omega}(\xi, \eta) = 0 \quad \text{at} \quad \xi = 0, \eta = 1 \qquad \ldots(3.14)$$

where ϕ is the conductance ratio.

Proceeding exactly along the same lines as in case I, we find that the solutions for the velocity and the magnetic field for laminar flow are

$$V_l = V_c \left[\epsilon R b_0 + (\epsilon R - \alpha) b_1 e^{\alpha\eta} + (\epsilon R - \beta) b_2 e^{\beta\eta} + b_3 + 1 - \epsilon R\eta\right] \qquad \ldots(3.15)$$

$$H_l = V_c R_M \left[b_0 + b_1 e^{\alpha\eta} + b_2 e^{\beta\eta} - \eta\right] \qquad \ldots(3.16)$$

and for turbulent flow

$$V_t = V_c \left[\epsilon R b_0 + (\epsilon R - \alpha) b_1 e^{\alpha\eta} + (\epsilon R - \beta) b_2 e^{\beta\eta} + b_3 + 1 - \epsilon R\eta + B_1 (\eta + 1)^2 + B_2 (\eta + 1)^n \qquad \ldots(3.17)$$

$$H_t = V_c R_M \left[b_0 + b_1 e^{\alpha\eta} b_2 e^{\beta\eta} - \eta + B_3 (\eta + 1)^2 + B_4 (\eta + 1)^n \qquad \ldots(3.18)$$

where $n > 2$ and

$$B_1 = -\frac{t-1}{2(\eta - 2)} \left[(\epsilon R - \alpha) b_1 \alpha e^{\alpha} + (\epsilon R - \beta) b_2 \beta e^{\beta} - \epsilon R \right]$$

$$B_2 = \frac{t-1}{2^{\eta-2}(\eta - 2)} \left[(\epsilon R - \alpha) b_1 \alpha e^{\alpha} + (\epsilon R - \beta) b_2 \beta e^{\beta} - \epsilon R \right]$$

$$B_3 = -\frac{(q-1)(\phi n + 2)}{2(\eta - 2)} \left[\alpha b_1 e^{\alpha} + \beta b_2 e^{\beta} - 1 \right]$$

$$B_4 = \frac{(q-1)(\phi n + 1)}{2^{\eta-1}(\phi + 1)} \left[\alpha b_1 e^{\alpha} + \beta b_2 e^{\beta} - 1 \right] \qquad \ldots(3.19)$$

$$b_0 = (\phi + 1) - (\phi\alpha + 1) b_1 e^{\alpha} - (\phi\beta + 1) b_2 e^{\beta}$$

$$b_1 = \frac{1}{\Lambda} \left[(\phi + 1)(\epsilon R - \beta) \sinh\beta - \epsilon R (\phi\beta \cosh\beta + \sinh\beta) \right]$$

$$b_2 = -\frac{1}{\Lambda}\left[(\phi + 1)(\epsilon R - \alpha)\sinh\alpha - \epsilon R(\phi\alpha\cosh\alpha + \sinh\alpha)\right]$$

$$b_3 = -\epsilon R b_0 - (\epsilon R - \alpha)\, b_1\, e^{\alpha} - (\epsilon R - \beta)\, b_2\, e^{\beta} - 1 + \epsilon R$$

$$\Lambda = (\epsilon R - \beta)\sinh\beta\,(\phi\alpha\cosh\alpha + \sinh\alpha)$$
$$\qquad - (\epsilon R - \alpha)\sinh\alpha\,(\phi\beta\cosh\beta + \sinh\beta)$$

and

$$t = \frac{\tau_t}{\tau_l} = \frac{\left(\dfrac{dV_t}{d\eta}\right)_{\eta=1}}{\left(\dfrac{dV_l}{d\eta}\right)_{\eta=1}}, \quad q = \frac{\left(\dfrac{dH_t}{d\eta}\right)_{\eta=1}}{\left(\dfrac{dH_t}{d\eta}\right)_{\eta=1}} = \frac{(H_t)_{\eta=1}}{(H_l)_{\eta=1}} \qquad \ldots(3.20)$$

are the empirical parameters. For turbulent flow $t, q > 1$ and for laminar flow $t, q = 1$.

3. Numerical Results

As the experimental results are not available, we assume for numerical discussions

$$R = 2\ ,\ M = 1\ ,\ R_M = 1.5\ ,\ \epsilon = 0.5\ ,\ A_0 = 1,$$
$$R^* = 3\ ,\ \phi = 0.5\ ,\ m = n = 3\ ,\ s = t = 3\ ,\ l = q = 4.$$

The behaviour of the velocity distribution has been shown in Figure 1 while that of the magnetic field has been exhibited in Figure 2. Continuous curves represent the nature of the entities for turbulent flow while the results for laminar flows are given by the dashed curves.

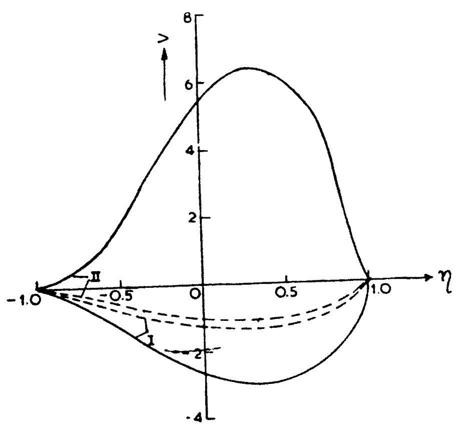

Fig. 1.

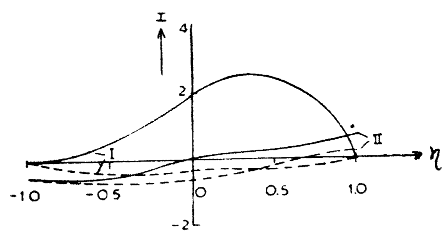

Fig. 2.

Figure 1 shows that the turbulence reduces the velocity for non-conducting walls, but it increases the velocity when the walls are conducting. On the other hand,

Figure 2 shows that the turbulence always increases the magnetic field whether the walls are conducting or non-conducting.

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