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ON A MONOTONICITY PROPERTY OF MEASURES OF DIRECTED-DIVERGENCE

J. N. KAPUR AND G. P. TRIPATHTI

Department of Mathematics, Indian Institute of Technology, New Delhi 110016

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For any three given probability distributions P, Q, R, it is shown that Csiszer's measure of directed divergence of $(kP + R)/(k + 1)$ from $(kQ + R)/(k + 1)$ is a monotonic increasing function of k.

1. INTRODUCTION

In Euclidean geometry, let P', Q' be points which divide the sides RP, RQ in the same ratio $k : 1$, then it is known that the length of $P'Q'$ is $k/(k + 1)$ times the length of PQ and $P'Q'$ is parallel to PQ.

Now let

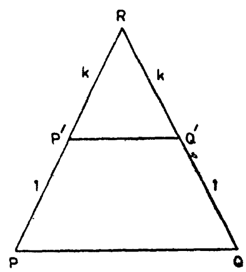

FIG. 1.

$$P = (p_1, p_2, ..., p_n),\ Q = (q_1, q_2, ..., q_n),\ R = (r_1, r_2, ..., r_n) \qquad ...(1)$$

be three probability distributions and let

$$P' = \frac{kp_1 + r_1}{k + 1}, \frac{kp_2 + r_2}{k + 1}, ..., \frac{kp_n + r_n}{k + 1}$$

$$Q' = \frac{kq_1 + r_1}{k + 1}, \frac{kq_2 + r_2}{k + 1}, ..., \frac{kq_n + r_n}{k + 1}. \qquad ...(2)$$

We now find directed divergences $D(P' : Q')$ and $D(P : Q)$, and motivated by the above result in Euclidean Geometry, ask the following questions :

(a) Is $D(P' : Q')/D(P : Q) = k/(k+1)$?

(b) Is $D(P' : Q') \leqq D(P : Q)$?

(c) Is $D(P' : Q')$ monotonic increasing function k?

Now there are a large number of measures of directed divergences[4]. The most important of these is the measure due to Csiszer[1], viz.

$$D(P: Q) = \sum_{i=1}^{n} q_i f\left(\frac{p_i}{p_i} \right) \qquad ...(3)$$

where $f(\,.\,)$ is a convex twice-differentiable function for which $f(1) = 0$. It can easily be shown that

$$D(P : Q) \geqslant 0 \qquad ...(4)$$

$$D(P : Q) = 0 \text{ iff } P = Q \qquad ...(5)$$

and

$D(P : Q)$ is a convex function of $p_1, p_2, ..., p_n$ as well as of $q_1, q_2, ..., q_n$.

The measure (3) includes the following measures of directed divergence as special cases:

(i) Kullback-Leibler[5] measure of directed divergence

$$D_1(P : Q) = \sum_{i=1}^{n} p_i \ln \frac{p_i}{q_i}. \qquad ...(6)$$

This is obtained by putting $f(x) = x \ln x$ in (3).

(ii) Havrda-Charvat[2] measure of directed divergence

$$D_2(P : Q) = \frac{1}{(\alpha - 1)}\left(\sum_{i-1} p^{\alpha}_{i} \; q_i^{1-\alpha} - 1 \right), \alpha > 0, \alpha \neq 1. \qquad ...(7)$$

This is obtained by putting $f(x) = (x^{\alpha} - x)/(\alpha - 1)$ in (3).

(iii) Sharma-Taneja[7] measure of directed divergence

$$D_3(P : Q) = \frac{1}{(\alpha - \beta)}\left(\sum_{i=1}^{n} p_i^{\alpha} \; q_i^{1-\alpha} - \sum_{i=1}^{n} p_i^{\beta} \; q_i^{1-\beta} \right) \qquad ...(8)$$

where

$$\alpha > 1,\ 0 < \beta < 1 \text{ or } 0 < \alpha < 1 \text{ and } \beta > 1. \qquad ...(9)$$

This is obtained by putting $f(x) = \frac{(x^{\alpha} - x^{\beta})}{(\alpha - \beta)}$ in (3). ...(10)

(iv) Kapur's[4] measure of directed divergence is

$$D_4(P:Q) = \sum_{i=1}^{n} p_i \ln \frac{p_i}{q_i} - \frac{b}{a^c} \sum_{i=1}^{n} (q_i + ap_i) \ln \frac{1 + a\frac{p_i}{q_i}}{1+a}$$

where $a^{c-1} > b$. ...(11)

This is obtained by putting

$$f(x) = x \ln x - \frac{b}{a^c}(1 + ax) \ln\left(\frac{1+ax}{1+a}\right) \quad \text{...(12) in (3).}$$

It is obvious that any theorem proved for Csiszer's measure (3) will continue to hold for the measures (6), (7), (8) and (11).

In the present paper, we shall show that for Csiszer's measure the answer to question (a) is in the negative, while the answers to questions (b) and (c) are in the affirmative. The answer to question (a) is also in the affirmative if equality sign there is replaced by sign $\leqslant$.

All the properties are highly desirable and are expected from every measure which seeks to measure discrepency or distance in some sense.

2. A Basic Theorem and Its Consequences

Theorem—Csizser's measure of directed divergence

$$D(P':Q') = D\left(\frac{kP+R}{k+1} : \frac{kQ+R}{k+1}\right) \quad \text{...(13)}$$

is a monotonic increasing function of k, ($k \geqslant 0$) and increases from 0 to $D(P:Q)$ as k increases from 0 to ∞.

Proof : Let

$$g(k) = D\left(\frac{kP+R}{k+1} : \frac{kQ+R}{k+1}\right)$$

$$= \sum_{i=1}^{n} \left(\frac{kq_i + r_i}{k+1}\right) f\left(\frac{kp_i + r_i}{kq_i + r_i}\right) \quad \text{...(14)}$$

so that

$$g'(k) = \frac{1}{(1+k)^2}\left[\sum_{i=1}^{n} (q_i - r_i) f\left(\frac{r_i + kp_i}{r_i + kq_i}\right)\right.$$

(equation continued on p. 854)

$$+ \sum_{i=1}^{n} \frac{(1+k)\, r_i\, (p_i - q_i)}{(r_i + kq_i)} f' \left(\frac{r_i + kp_i}{r_i + kq_i} \right) \Big] \quad \ldots(15)$$

$$= \frac{R\ (k)}{(1+k)^2} \text{ (say).} \quad \ldots(16)$$

Now

$$R'(k) = \sum_{i=1}^{n} \frac{(1+k)\ r_i^2\ (p_i - q_i)^2}{(r_i + kq_i)^3} f'' \left(\frac{r_i + kp_i}{r_i + kq_i} \right) \quad \ldots(17)$$

since $f(.)$ is convex, $f''(.) \geqslant 0$ and $R'(k)$ vanishes when $p_i = q_i$ for each i so that $R'(k) \geqslant 0$. Also from (15) and (16), $R(0) = (0)$, since $f(1) = 0$ and $\sum_{i=1}^{n} (p_i - q_i) = 0$ so. Thus $R(k) \geqslant 0$. Thus $g'(k) \geqslant 0$ so that $g(k)$ is a monotonic increasing function of k. Also $g(0) = 0$ and $g(\infty) = D(P : Q)$. Thus the theorem is proved.

We deduce the following corollaries from this theorem.

Corollary 1—The answers to questions (b) and (c) of section 1 are in the affirmative.

Corollary 2—For each of the measures due to Kullback-Leibler[5], Havrda-Charvat[2], Sharma-Taneja and Kapur[4], the directed divergence of $\frac{(kP+R)}{(k+1)}$ from $\frac{(kQ+R)}{(k+1)}$ is a monotonic increasing function of k which increases from 0 to the the value of directed divergence of P from Q, as k increases from 0 to ∞.

Corollary 3—For Renyi's measure of directed divergence defined by

$$D_5(P : Q) = \frac{1}{(\alpha - 1)} \ln \sum_{i=1}^{n} p_i^{\alpha}\ q_i^{1-\alpha},\ \alpha \neq 1,\ \alpha > 0 \quad \ldots(18)$$

$D_5 \left[\frac{(kP+R)}{(k+1)} : \frac{(kQ+R)}{(k+1)} \right]$ is a monotonic increasing function of k which increases from 0 to $D_5(P : Q)$ as k increases from 0 to ∞.

PROOF: From (7) and (18)

$$D_5(P : Q) = \frac{1}{(\alpha - 1)} \ln [(\alpha - 1)\, D_2(P : Q) + 1] \quad \ldots(19)$$

so that

$$D_5(P' : Q') = \frac{1}{[(\alpha - 1)\, D_2(P' : Q') + 1]} \frac{d}{dk} D_2(P' : Q')$$

(*equation continued on p.* 855)

$$= \frac{1}{\sum_{i=1}^{n} p_i'^{\alpha} \; q_i'^{1-\alpha}} \; \frac{d}{dk} \; D_2 (P' : Q'). \qquad \text{...(20)}$$

Since $\sum_{i=1}^{n} p_i^{\alpha} \quad q_i^{1-\alpha} > 0$ and $D_2 \; (P' : Q')$ is a monotonic increasing function of k it follows from (20) that $D \, (P' : Q')$ is also a monotonic increasing function of k.

Corollary 4—For Kapur's[3] measare of directed divergence defined by

$$D_6 \, (P : Q) = \frac{1}{(\alpha - \beta)} \; \ln \; \frac{\sum_{i=1}^{n} p_i^{\alpha} \;\; q_i^{1-\alpha}}{\sum_{i=1}^{n} p_i^{\beta} \; q_i^{1-\beta}} \; ; \quad \begin{array}{l} 0 < \alpha < 1, \beta > 1, \text{or} \\ 0 < 1 < i, \alpha < 1. \end{array} \qquad \text{...(21)}$$

$D_6 \; (P' : Q')$ is a monotonic increasing function of k which increases from 0 to D_6 $(P : Q)$ as k increases from 0 to ∞.

PROOF :

$$D \, (P' : \; Q') = \frac{\alpha - 1}{\alpha - \beta} \left[\frac{1}{\alpha - 1} \ln \sum_{i=1}^{n} p_i'^{\alpha} \cdot q_i'^{1-\alpha} \right]$$

$$+ \; \frac{1 - \beta}{\alpha - \beta} \left[\frac{1}{\beta - 1} \; \ln \sum_{i=1}^{n} p_i^{\beta} \;\; q_i^{1-\beta} \; \right]. \qquad \text{...(22)}$$

From Corollary 3, the expressions within the square brackets are monotonic increasing functions of k. Also if the conditions on α, β given in (21) are satisfied, then the coefficients of both the expressions in square brackets are positive. It follows that $D_6 \; (P' : Q')$ is a monotonic increasing function of k.

Corollary 5—: The answers to questions (b) and (c) of section 1 are in the affirmative for both Renyi's[6] and Kapur'[3] second measure of directed divergence.

3. Discussion of Question (a) of Section 1

Since each of the measures of directed divergence due to Csiszer[1], Kullback-Leibler[5], Havrda-Charvat[2], Sharma-Taneja[7] and Kapur[4] is a convex function of both P and Q, it follows that for each of these

$$D[mP + (1 - m) R : mQ + (1 - m) R]$$
$$\leqslant mD (P : Q) + (1 - m) D (R : R) = mD (P : Q), 0 \leqslant m \leqslant 1. \quad \ldots(23)$$

Putting $m = \frac{k}{k+1}$, we get

$$D\left(\frac{kP + R}{k + 1} : \frac{kQ + R}{k + 1} \right) \leqslant \frac{k}{(k + 1)} D (P : Q) \quad \ldots(24)$$

or

$$D (P' : Q') \leqslant \frac{k}{(1 + k)} D (P : Q). \quad \ldots(25)$$

Thus the answer to question (a) for all these measures is in the affirmative if the equality sign there is replaced by $\leqslant$ sign.

By taking special probability distributions, it can easily be shown that in (25) equality sign does not always hold for Csiszer's measure or for any of its special cases.

The equality sign will however hold for the metrics

$$D (P : Q) = [\sum_{i=1}^{n} (p_i - q_i)^{2r})^{1/2_r}, \quad r = 1, 2, 3 \quad \ldots(26)$$

or

$$D (P : Q) = [\sum_{i=1}^{n} | p_i - q_i | ^r]^{1/r}, \quad r = 1, 2, 3, \ldots \quad \ldots(27)$$

It may be noted that while these metrics are symmetric and satisfy the triangle inequality, our directed divergence measures do not satisfy these properties.

If we take symmetric measures of directed divergence defined by

$$J (P : Q) = D (P : Q) + D (Q : P) \quad \ldots(28)$$

questions (b) and (c) are still answered in the affirmative, while question (a) is answered in the affirmative if we replace equality sign by inequality sign.

4. An Interesting Property of Symmetric Diverwence

When $D (P : Q)$ is a convex function of both P and Q, we get

$$D \left(\frac{kP + Q}{k + 1} : R \right) \leqslant \frac{k}{(k + 1)} D (P : R) + \frac{1}{(k + 1)} D (Q : R) \quad \ldots(29)$$

$$D \left(R : \frac{kP + Q}{k + 1}\right) \leqslant \frac{k}{(k + 1)} D (R : P) + \frac{1}{(k + 1)} D (R : Q) \quad \ldots(30)$$

so that

$$J\left(\frac{kP+Q}{k+1}:R\right) \leqslant \frac{k}{(k+1)} J(P:R) + \frac{1}{(k+1)} J(Q:R). \quad \ldots(31)$$

Similaraly

$$J\left(\frac{kQ+R}{k+1}:P\right) \leqslant \frac{k}{(k+1)} J(Q:P) + \frac{1}{(k+1)} J(R:P) \quad \ldots(32)$$

and

$$J\left(\frac{kR+P}{k+1}:Q\right) \leqslant \frac{k}{(k+1)} J(R:Q) + \frac{1}{(k+1)} J(P:Q). \quad \ldots(33)$$

From (31) – (33) we get

$$J\left(\frac{kP+Q}{k+1}:R\right) + J\left(\frac{kQ+R}{k+1}:P\right) + J\left(\frac{kR+P}{k+1}:Q\right)$$
$$\leqslant J(P:Q) + J(Q:R) + J(R:P). \quad \ldots(34)$$

In particular if $k = 1$,

$$J\left(\frac{P+Q}{2}:R\right) + J\left(\frac{Q+R}{2}:P\right) + (J\left(\frac{R+R}{2}:Q\right)$$
$$\leqslant J(P:Q) + J(Q:R) + J(R:P). \quad \ldots(35)$$

The inequalities (34) and (35) will hold for both Euclidean metrics as well as for Csiszer's symmetric divergence and its special cases.

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