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ON COMPLETE INTEGRAL CLOSURE OF G-DOMAIN

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Let R be a commutative ring with identity 1 and let R' be the total quotient ring of R. It is well known that the complete integral closure R_0 of R in R' is a subring of R'. However R_0 is not necessarily itself completely integrally closed. We denote by $R^{(1)} = R_0$, $R^{(2)} = (R_0)_0 = R_{00}$ and for positive integer $n \geqslant 3$, $R^{(n)} = (R^{(n-1)})_0$. As pointed out by Professor Silvio Greco, it will be quite interesting to find classes of rings R for which there exists an integer $n \geqslant 1$ such that $R^{(n)} = R^{(n+1)}$ and hence $R^{(n)} = R^{(m)}$ for all $m \geqslant n$. For Noetherian domain R, it is well known that $R^{(1)} = R^{(2)}$. In this paper we show that if R is a G-domain then $R^{(2)} = R^{(3)}$, i.e. $R_{00} = R_{000}$ (Note that there is a G-domain R for which $R^{(1)} \neq R^{(2)}$ i.e $R_0 \neq R_{00}$).

1. ALMOST INTEGRAL DEPENDENCE

Let R be a commutative ring with identity 1. Let R' be the total quotient ring of R. An element $a \in R'$ is called almost integral over R if there exists a finitely generated submodule of the R-module R' which contains all powers of a. The ring R_0 of elements of R' which are almost integral over R is called the complete integral closure of R in R'. If $R_0 = R$, then R is called completely integrally closed. Some important results on complete integral closure have been proved by Gilmer and Heinzer[2]. The following theorem is useful for determining the complete integral closure of a ring.

Theorem 1.1—(Larsen and McCarthy[6], Theorem 4.20)—The complete integral closure of a ring R with total quotient ring R' is

$$R_0 = \left[\begin{array}{l} x : x \in R' \text{ and there exists a regular element } r \in R \\ \text{such that } rx^n \in R \text{ for } n = 0, 1, 2, \ldots \end{array} \right]$$

It is well known that R_0 is not necessarily completely integrally closed. (Larsen and McCarthy[6], Ch. IV, Ex. 14).

2. G-DOMAINS

An integral domain R with quotient field K is called a G-domain if it satisfies one of the following equivalent conditions

1. K is a finitely generated R-algebra
2. K can be generated by one element as an R-algebra.

These rings were first introduced and studied independently by Goldman[3] and Krull[5] at the same time. Later Noetherian G-domains were studied by Artin and Tate[1]. See also Kaplansky[4], 1.3.

It is shown (Singh[7]) that there is a G-domain R such that R_0 is not completely integrally closed i. e. $R_0 \neq R_{00}$.

The following theorem shows that if R is G-domain then R_{00} is completely integrally closed.

Theorem 2.2—If R is a G-domain, then $R_{000} = R_{00}$.

We prove this theorem in a series of lemmas.

Lemma 1—Let R be a G-domain with quotient field K. If $x \in K$ be such that $x^k \in R_0$ for some positive integer k, then $x \in R_0$.

PROOF : Let $K = R\,[u^{-1}]$. There exist positive integers $t_1\ t_2, \ldots, t_{k-1}$ such that $x^i \in u^{-t_i} R$ for $i = 1, 2, \ldots, k-1$.

Put $t = \max\{t_1, t_2, \ldots, t_{k-1}\}$. Therefore $u^t x^i \in R$ for $i = 1, 2, \ldots, k-1$. Since $x^k \in R_0$, there exists a non-zero element c in R such that $c\,(x^k)^n \in R$ for $n = 0, 1, 2, \ldots$. Writing each positive integer m as $m = kn + r$ where $0 \leqslant r \leqslant k-1$, we have $(u^t c)\ x^m = (u^t\ c).\ x^{kn+r} = (u^t\ x^r)\ c\,(x^k)^n \in R$. $R \subseteq R$. Therefore $x \in R_0$.

Lemma 2—Let R be a G-domain with quotient field K and let $K = R\ [u^{-1}]$. Let $N = \{x \in K : x^n \in uR$ for some positive integer $n\}$ and $T = \{x \in K : x\,N \subseteq N\}$. Then T is a subring of K containing R, and hence $R_{00} \subseteq T_{00}$.

PROOF : First we observe that if $R = K$, then $R = N = T = K$ and there is nothing to prove. Suppose $R \neq K$. In this case one can easily verify that ... $u^2 R \underset{\neq}{\subset} uR \underset{\neq}{\subset} R \underset{\neq}{\subset} u^{-1} R \underset{\neq}{\subset} u^{-2} R \ldots$ and $K = \bigcup_{n=1}^{\infty} u^{-n} R$. We now prove that N is an R-submodule of K. It is clear that $0 \in N$ and that $x \in N$, $a \in R$ implies $a\,x \in N$. Now suppose $x, y \in N$. We can find a positive integer n such that $x^n \in u\,R$ and $y^n \in uR$. Choose a positive integer k such that $x, x^2, \ldots x^{n-1}, y, y^2, \ldots, y^{n-1} \in u^{-k} R$. Now every positive integer m can be written as $m = qn + r$ where $0 \leqslant r \leqslant n-1$. Therefore $x^m = (x^n)^q.\ x^r \in (uR)^q\, u^{-k} R \subseteq u^{-k+1} R$. Similarly $y^m \in u^{-k+1} R$. Take $n_0 = 2kn$. For $j \geqslant n_0$, writing $j = tn_0 + s$ where $0 \leqslant s \leqslant n_0 - 1$, we have $x^j = (x^n)^{2kt}\, x^s \in (u^{2kt} R)\,(u^{-k+1} R) \subseteq u^k R$ and $y^j \in u^k R$. Therefore $(x + y)^{2n}. O = \sum_{i+j=2n_0} x^i y^j \in \Sigma\ (u^{-k+1} R)\,(u^k R) \subseteq u\,R$ and hence $x + y \in N$.

Now since N is an R-submodule of K, it is clear that T is a subring of K containing R.

Lemma 3—Let R be a G-domain with quotient field K and let T be as defined in Lemma 2; Then $T \subseteq R_{00}$.

PROOF : Let $x \in T$. Note that $u \neq 0$, $u \in N$. By definition of T, we have $xu \in N$. and so there exists a positive integer n such that $(xu)^n \subset uR \subseteq R \subseteq R_0$.

Using Lemma 1, we have $xu \in R_0$. Since $x \in T$ implies $x^n \in T$ for every positive integer n, by the same argument we get $ux^n \in R_0$ for $n = 0, 1, 2, \ldots$ and hence $x \in R_{00}$. Since x was an arbitrary element of T, we get $T \subseteq R_{00}$.

Lemma 4—Let R be a G-domain with quotient field K and T be subring of K as defined in Lemma 2; then T is completely integrally closed. In particular $T = T_{00}$.

PROOF : Let $x \in T_0$. There exists a non-zero element c in T such that $cx^n \in T$ for $n = 0, 1, 2, \ldots$. Since $K = R[u^{-1}]$ there exists a positive integer k such that $c^{-1} \in u^{-k} R$. Choose an arbitrary element y in N and fix it. There exists a positive integer t such that $y^t \in uR$; therefor $y^{t(k+1)} \in u^{k+1} R$ and hence $c^{-1} y^{t(k+1)} \in uR \subseteq N$; and since $cx^n \in T$, this implies $x^n y^{t(k+1)} \in N$ for $n = 0, 1, 2, \ldots$

Taking $n = t(k+1)$ we get that $(xy)^{t(k+1)} \in N$ and hence $xy \in N$. Since y was an arbitrary element of N, we get that $x \in T$. Therefoie $T_0 \subseteq T$, i. e T is completely integrally closed.

Proof of Theorem 2.2—By Lemma 2 and Lemma 4, $R_{00} \subseteq T$. Using Lemma 3, we get that $R_{00} = T$. Now using Lemma 4 again, we get $R_{000} = R_{00}$, i.e R_{00} is completely integrally closed.

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