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ON ALMOST UNIFIED CONTACT FINSLER STRUCTURES AND CONNECTIONS

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The purpose of the present paper is to unify the almost contact Finsler structure[2] and almost para contact Finsler structure[1,3] on a differentiable manifold M, and to determine all Finsler connections compatible with the unified structure.

1. INTRODUCTION

Let M be an n-dimensional C^∞-differentiable manifold equipped with a Finsler connection $F\Gamma = (F, N, C)$ in the sense of Matsumoto[4]. Then the h- and v-covariant derivatives of a Finsler tensor field K of type (1.1) is given below :

$$\left.\begin{aligned} K^i_{j|k} &\quad \delta K^i_j/\delta x^k + F^i_{mk} K^m_j - F^m_{jk} K^i_m \\ K^i_{j|k} &= \partial K^i_j/\partial y^k + C^i_{mk} K^m_j - C^m_{jk} K^i_m \end{aligned}\right\} \qquad \ldots(1.1)$$

where

$$\delta/\delta x^k = \partial/\partial x^k - N^i_k (\partial/\partial y^i).$$

If $F\Gamma = (F, N, C)$ and $F\bar{\Gamma} = (\bar{F}, \bar{N}, \bar{C})$ are two Finsler connections on M, then the transformation from $F\Gamma$ to $F\bar{\Gamma}$ is given by:

$$\left.\begin{aligned} \bar{N}^i_j &= N^i_j - A^i_j \\ \bar{F}^i_{jk} &= F^i_{jk} + C^i_{jh} A^h_k - B^i_{jk} \\ \bar{C}^i_{jk} &= C^i_{jk} - D^i_{jk} \end{aligned}\right\} \qquad \ldots(1.2)$$

where A, B, D are the difference tensor fields[4].

2. ALMOST UNIFIED CONTACT FINSLER STRUCTURES

An almost unified contact Finsler structure on M of odd dimension is defined by a triad (ϕ, η, ξ) of a Finsler tensor field $\phi(x, y)$ of type (1,1), a l-form $\eta(x, y)$ and a vector field $\xi(x, y)$ satisfying the following conditions :

$$\left.\begin{aligned} &\phi_j^a \phi_a^k = \mu \left\{ \delta_j^k - \eta_j \xi^k \right\} ; \text{rank} \left(\phi_j^a \right) = n - 1 \\ &\phi_j^a \eta_a = 0 \\ &\phi_j^a \xi^j = 0 \\ &\eta_a (\xi^a) = 1 \end{aligned}\right\} \quad \ldots(2.1)$$

where μ is some non-zero constant.

Remark : When $\mu = +1$, the structure (ϕ, η, ξ) defined above becomes an almost para-contact structure on M and when $\mu = -1$, it becomes an almost contact Finsler structure on M.

Analogous to Obata operators[6], we have,

$$\left.\begin{aligned} &\underset{1}{0} = 1/2\,(I \otimes I + 1/\mu\,(\phi \otimes \phi)) \\ &\underset{2}{0} = 1/2\,(I \otimes I - 1/\mu\,(\phi \otimes \phi)) \\ &\Omega = 1/2\,(\eta \otimes \xi \otimes I + I \otimes \eta \otimes \xi - \eta \otimes \xi \otimes \eta \otimes \xi) \end{aligned}\right\} \quad \ldots(2\ 2)$$

where I is the identity operator.

The operators act, for example, in the following way:

$$(\underset{1}{0}.\ t)^k_{mj} = \underset{1}{0}{}^{kh}_{ij}\ t^i_{hm} \quad (\underset{1}{0},\ \Omega)^{kb}_{mj} = \underset{1}{0}{}^{kh}_{ij}\ \Omega^{ib}_{mh} \quad \ldots(2.3)$$

where t is an arbitrary Finsler tensor field of type (1, 2). They satisfy the following relations.

$$\text{(a)}\quad \underset{1}{0} + \underset{2}{0} = I \otimes I\ ;\ \underset{2}{0}.\ \underset{1}{0} = \underset{1}{0}.\ \underset{2}{0} = \underset{2}{0}.\ \Omega = \Omega\ .\underset{2}{0}$$

$$= \underset{1}{0}\,\Omega = \Omega.\ \underset{1}{0} = \Omega.\ \Omega = \Omega/2$$

$$\text{(b)}\quad (\underset{2}{0} + \Omega)\,(\underset{1}{0} - \Omega) = (\underset{1}{0} - \Omega)\ (\underset{2}{0} + \Omega) = 0.0. \quad \ldots(2.4)$$

If we put $\underset{1}{Q} = \underset{1}{0} - \Omega$ and $\underset{2}{Q} = \underset{2}{0} + \Omega$, then (2.4b) becomes

$$\underset{2}{Q}.\ \underset{1}{Q} = \underset{1}{Q}.\ \underset{2}{Q} = 0.\ 0$$

Lemma 2.1—A system of tensor equations $\underset{2}{Q} X = A$ with X as unknowns has solutions if and only if $\underset{1}{Q} A = 0$. Then the general solution of $\underset{2}{Q}. X = A$ is,

$$X = A + \underset{1}{Q} \cdot Y$$

where Y is an arbitrary Finsler tensor field of the same type as that of X.

3. Finsler Connections Compatible with Almost Unified Contact Finsler Structure

Definition 3.1—A Finsler connection $F\Gamma = (F, N, C)$ is called an almost unified contact Finsler connection relative to (ϕ, η, ξ) if :

$$\left.\begin{array}{l} \phi^i_{j|k} = 0, \phi^i_{j\|k} = 0, \ \eta_{i|k} = 0, \eta_{i\|k} = 0 \\ \xi^i_{|k} = 0, \xi^i_{\|k} = 0. \end{array}\right] \qquad \ldots(3.1)$$

It can be shown that for any almost unified contact Finsler connection the operators $\underset{1}{0}$, $\underset{2}{0}$, $\underset{1}{Q}$, $\underset{2}{Q}$ and Ω are h- and v-covariant constants.

Giving a Finsler connection $F\overset{0}{\Gamma} = (\overset{0}{F}, \overset{0}{N}, \overset{0}{C})$ on M, any Finsler connection $F\Gamma = (F, N, C)$ on M can be expressed in terms of the difference tensors X^i_j , B^i_{jk} and D^i_{jk} as :

$$\left.\begin{array}{l} N^i_j = \overset{0}{N}{}^i_j - X^i_j \\ F^i_{jk} = \overset{0}{F}{}^i_{jk} + \overset{0}{C}{}^i_{jm} X^m_k - B^i_{jk} \\ C^i_{jk} = \overset{0}{C}{}^i_{jk} - D^i_{jk} . \end{array}\right\} \qquad \ldots(3.2)$$

In order that $F\Gamma = (F, N, C)$ be an almost unified contact Finsler connection, we obtain by Lemma (2.1) as :

$$\begin{aligned} B^i_{jk} = 1/2 \Big\{ &- \xi^i \Big(\eta_j \eta_h \xi^{h}{}^{0}_{\downarrow k} + \xi^{h}{}^{0}_{| a} X^a_k + \eta_j {}^{0}_{| k} \\ &+ \eta_j {}^{0}_{| a} X^a_k \Big) + 2\eta_j \Big(\xi^{i}{}^{0}_{\downarrow k} + \xi^{i}{}^{0}_{| a} X^a_k \Big) \\ &+ 1/\mu\, \phi^h_j \Big(\phi^i_h {}^{0}_{| k} + \phi^i_h {}^{0}_{| a} X^a_k \Big) \Big\} + \underset{1}{Q}{}^{ih}_{aj} Y^a_{hk} \end{aligned}$$

$$D^i_{jk} = 1/2\left\{-\xi^i\left(\eta_j\eta_h\,\xi^{h0}_{\ \ |k} + \eta_{j\ |k}^{\ 0}\right) + 2\eta_i\,\xi^{i0}_{\ \ |k} + 1/\mu\,\phi^h_j\,\phi^{i\ 0}_{h\ |k}\right\} + \underset{1}{Q}{}^{ih}_{aj}\,Z^a_{hk}$$

where Y and Z are arbitrary Finsler tensor field of type (1,2) (cf. Miron and Hashiguchi[5]).

Consequently, we have,

Theorem 3.1—The general family of the almost unified contact Finsler connections $F\Gamma = (F, N\ C)$ relative to an almost unified contact Finsler structure (ϕ, η, ξ) is given by :

$$\text{(a)}\quad N^i_j = \overset{0}{N}{}^i_j - X^i_j$$

$$\text{(b)}\quad F^i_{jk} = \overset{0}{F}{}^i_{jk} + \overset{0}{C}{}^i_{ja}\,X^a_k - 1/2\Big\{-\xi^i\big(\eta_j\eta_h\,\xi^{h0}_{\ \ \downarrow k} + \xi^{h0}_{\ \ |a}\,X^a_k + \eta_{j\ |k}^{\ 0} + \eta_{j\ |a}^{\ 0}X^a_k\big) + 2\eta_j\left(\xi^{i0}_{\ \ \downarrow k} + \xi^{i0}_{\ \ |a}\,X^a_k\right) + 1/\mu\phi^h_j\left(\phi^{i0}_{h\ |k} + \phi^{i\ 0}_{h\ |a}\,X^a_k\right)\Big\} - \underset{1}{Q}{}^{ih}_{aj}\,Y^a_{hk}$$

$$\text{(c)}\quad C^i_{jk} = \overset{0}{C}{}^i_{jk} - 1/2\left\{-\xi^i\left(\eta_j\eta_h\,\xi^{h0k}_{\ \ |} + \eta^{j0}_{\ \ |k}\right) + 2\eta_j\,\xi^{i0}_{\ \ |k} + 1/\mu\,\phi^h_j\,\phi^{i\ 0}_{h\ |k}\right\} - \underset{1}{Q}{}^{jk}_{aj}\,Z^a_{hk}\,. \qquad \ldots(3.3)$$

Particularly by putting $X^i_j = 0$, $Y^i_{jk} = 0$ and $Z^i_{jk} = 0$ in (3.3), we have,

Theorem 3.2—If the initial Finsler connection is $\overset{0}{F\Gamma} = (\overset{0}{F}, \overset{0}{N}, \overset{0}{C})$ then the following Finsler connection

$$\text{(a)}\quad \overset{k}{N}{}^i_j = \overset{0}{N}{}^i_j$$

$$\text{(b)}\quad \overset{k}{F}{}^i_k = \overset{0}{F}{}^i_{jk} - 1/2\left\{-\xi^i\left(\eta_j\eta_h\,\xi^{h0}_{\ \ |k} + \eta_{j\ |k}^{\ 0}\right) + 2\,\eta_j\,\xi^{i0}_{\ \ |k} + 1/\mu\,\phi^h_j\,\phi^{i\ 0}_{h\ |k}\right\}$$

$$\text{(c)} \quad \overset{k}{C}{}^{i}_{jk} = \overset{0}{C}{}^{i}_{jk} - 1/2\left\{ - \xi^{i}\left(\eta_j \eta_h \xi^{h0}{}_{|k} + \eta_j {}_{|}{}^{0}{}_{k}\right) + 2\, \eta_j \xi^{i0}{}_{|k} + 1/\mu\, \phi^{h}_{j}\, \phi^{i\,0}_{h\,|k} \right\} \qquad ..(3.4)$$

is an almost unified contact Finsler connection

Definition 3.2—The Finsler connection $F\overset{k}{\Gamma} = (\overset{k}{F}, \overset{k}{N}, \overset{k}{C})$ given by (3.4) is called the almost unified Kawaguchi connection derived from $F\overset{0}{\Gamma} = (\overset{0}{F}, \overset{0}{N}, \overset{0}{C})$.

Now, let $F\overset{m}{\Gamma} = (\overset{m}{F}, \overset{m}{N}, \overset{m}{C})$ be the Matsumoto connection then,

$$\overset{m}{N}{}^{i}_{j} = \overset{0}{N}{}^{i}_{jk},\ \overset{m}{F}{}^{i}_{jk} = \partial \overset{0}{N}{}^{i}_{k} / \partial y^{j}\ \overset{m}{C}{}^{i}_{jk} = \overset{0}{C}{}^{i}_{jk}.$$

Replacing the initial Finsler connection $F\overset{0}{\Gamma} = (\overset{0}{F}, \overset{0}{N}, \overset{0}{C})$ by the Matsumoto connection in (3.4), we get an almost unified contact Finsler connection denoted by $F\overset{q}{\Gamma} = (\overset{q}{F}, \overset{q}{N}, \overset{q}{C})$ called an almost unified canonical contact Finsler connection.

Theorm 3.3—If we replace the initial Finsler connection $F\overset{0}{\Gamma} = (\overset{0}{F}, \overset{0}{N}, \overset{0}{C})$ by an almost unified contact Finsler connections $F\Gamma = (F, N, C)$ then the general family of the almost unified contact Finsler connection $F\bar{\Gamma} = (\bar{F}, \bar{N}, \bar{C})$ is given by :

$$\bar{N}^{i}_{j} = N^{i}_{j} - X^{i}_{j}$$

$$\bar{F}^{i}_{jk} = F^{i}_{jk} + C^{i}_{ja}\, X^{a}_{k} - \underset{1}{Q}{}^{ih}_{aj}\, Y^{a}_{hk}$$

$$\bar{C}^{i}_{jk} = C^{i}_{jk} - Q^{ih}_{aj}\, Z^{a}_{hk}. \qquad ...(3.5)$$

The Finsler connections having the non-linear connection N common is denoted by $F\Gamma(N)$.

Theorem 3.4—The family of the almost unified contact Finsler connections $F\ \bar{\Gamma}(N)$ is given by :

$$\text{(a)} \quad \bar{N}^{i}_{j} = N^{i}_{j}$$

$$\text{(b)} \quad \bar{F}^{i}_{jk} = F^{i}_{jk} - \underset{1}{Q}{}^{ih}_{aj}\, Y^{a}_{hk}$$

$$\text{(c)} \quad \bar{C}^{i}_{jk} = C^{i}_{jk} - \underset{1}{Q}{}^{ih}_{aj}\, Z^{a}_{hk} \qquad ...(3.6)$$

where Y and Z are arbitrary Finsler tensor fields of type (1, 2).

We notice that (3.6) determine the transformation $F(N) \rightarrow F\bar{\Gamma}(N)$ of the almost unified contact Finsler connections having a common non-linear connection N.

Theorem 3.5—The set of all transformations of the almost unified contact Finsler connections given by (3.6) is an abelian group with respect to the product of mapping and which is isomorphic to the additive group of the pairs of the Finsler tensor fields $(\underset{1}{Q}.\, Y;\ \underset{1}{Q}.\, Z)$.

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REFERENCES

1. Gh. Atanasiu, *Analele Univ.*, *'Al. I. Cuza, Iași,* **4** (1983), 1-4.
2. Gh. Atanasiu, B. B. Sinha and S. K. Singh, *Proc. Nat. Semi. on Finsler Spaces, Brasov,* 1984, pp. 29-36.
3. A. Bucki and A. Miesnowski, *Acta Math. Hung.* **43/3, 4** (1985), 327-36.
4. M. Matsumoto, *Foundations of Finsler Geometry and Special Finsler Spaces,* Kaiseisha Press, Otsu, Japan, 1986.
5. R. Miron and M. Hashiguchi, *Rep. Fac. Sci. Kagoshima Univ.* **12** (1979), 21-25.
6. V. Oproiu. *Rev. Roumaine Math. Pures Appl.* **1161** (1971), 383-93.