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ON F-ABSOLUTELY TRANSLATIVE SUMMABILITY METHODS

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We call the matrix $A = (a_{nk})$ F-regular if it transforms F into F and f-lim $A(x) = f$-lim x for all x belonging to F where F denotes the set of all almost convergent sequences. We write $A \sim A'$ to denote that the F-regular matrices A and A' are F-absolutely equivalent for all bounded sequences. In the present paper we show that $AB \sim A'B'$ if A, A', B, B' are F-regular and if $A \sim A'$ and $B \sim B'$ (where AB denotes the composition product). We also show that an F-regular matrix is always F-absolutely translative for all bounded sequences.

1. INTRODUCTION

Let l_∞ and c be the Banach spaces of bounded and convergent sequences with the usual supremum norm.

Let $A = (a_{nk})$ be an infinite matrix and $x = (x_k)$ be sequence with complex terms. If the sequence

$$(A_n(x)) = \left(\sum_{k=0}^{\infty} a_{nk} x_k\right)$$

exists (i. e. the series on the right hand side converges for each n) then the sequence $A(x) = (A_n(x))$ is called the A-transform of $x = (x_k)$. The matrix $A = (a_{nk})$ is said to be regular if the A-transform of x is convergent to the limit of x for each $x \in c$. The regularity conditions of A are well known[1].

It is shown in Lorentz[3] that a sequence $x = (x_n) \in l_\infty$ is almost convegent to s if and only if

$$\lim_{q} \frac{1}{q+1} \sum_{i=0}^{q} x_{n+i} = s, \text{ uniformly in } n. \qquad \ldots(1)$$

We write f-lim $x = s$ whenever (1) holds. By F we denote the linear space of all almost convergent sequences. It is called that a sequence x is almost A-summable to L if f-lim $A(x) = L$.

We call the matrix $A = (a_{nk})$ is F-reqular if it transforms F into F and f-lim $A(x) = f$-lim x for each $x \in F$. The following theorem gives the necessary and sufficient conditions for a matrix to be F-regular.

Theorem 1.1—A matrix $A = (a_{nk})$ is F-regular if and only if

(i) $\sup_n \Sigma_k \mid a_{nk} \mid < \infty$

(ii) f-lim $a_{nk} = 0$ for each k

(iii) f-lim $\Sigma_k \; a_{nk} = 1$

$$\text{(iv)} \quad \lim \sum_{k=0}^{\infty} \frac{1}{q+1} \left| \sum_{i=0}^{q} (a_{n+i,k} - a_{n+i,k+1}) \right| = 0$$

uniformly in n (Duran[2]).

Throughout the paper, the sums will be taken from $k = 0$ to $k = \infty$.

2. F-Absolute Equivalence of Summability Methods

Concerning with the F-absolute equivalence of F-regular summability methods we have the following definition :

Definition 2.1—Let $A = (a_{nk})$ and $B = (b_{nk})$ be two F-regular matrices and $x = (x_k)$ be a sequence for which

$$z'_n = \Sigma_k \, a_{nk} \, x_k \quad \text{and} \quad z''_n = \Sigma_k \, b_{nk} \, x_k$$

exist.

Then A and B are said to be F-absolutely equivalent for a given class of sequences (x_k) if

$$f\text{-}\lim \left(z'_n - z''_n \right) = 0$$

i.e. either $\left(z'_n \right)$ and $\left(z''_n \right)$ both almost converge to the same value, or else neither of them almost converges but their difference almost converges to zero[4].

The following theorem is known.

Theorem 2.2—Let A and B be two F-regular matrices. Then A and B are F-absolutely equivalent for all bounded sequences if and only if

$$\lim \sum_{k=0}^{\infty} \frac{1}{1+q} \left| \sum_{i=0}^{q} (a_{n+i,k} - b_{n+i,k}) \right| = 0$$

uniformly in n (Orhan[4]).

3. F-Absolute Equivalence of Composition Products (AB) and (BA)

The composition products (AB) and (BA) are respectively given by

$$v_n = \sum_{p=0}^{\infty} \left(\sum_{k=0}^{\infty} a_{nk} \, b_{kp} \right) z_p = \sum_{p=0}^{\infty} \sum_{k=0}^{\infty} a_{nk} \, b_{kp} \, z_p \qquad \ldots(2)$$

and

$$v'_n = \sum_{p=0}^{\infty} \left(\sum_{k=0}^{\infty} b_{nk} \, a_{kp} \right) z_p = \sum_{p=0}^{\infty} \sum_{k=0}^{\infty} b_{nk} \, a_{kp} \, z_p. \qquad \ldots(3)$$

If A, B are F-regular and (z_p) is bounded then, by Theorem 1.1, the double sums in (2) is absolutely convergent, so that inversion in the order of summation is justified and therefore

$$\sum_{p=0}^{\infty} z_p \sum_{k=0}^{\infty} a_{nk} \, b_{kp} = \sum_{k=0}^{\infty} a_{nk} \sum_{p=0}^{\infty} b_{kp} \, z_p$$

for all n; in other words

$$(A\,B)\,z = A\,(Bz). \qquad \ldots(4)$$

That is to say the result of transforming the sequence z by the product AB is the same as transforming z by the matrix B and then transforming the result by the matrix A.

It is known that (AB) and (BA) are absolutely equivalent for all bounded sequences if the T-matrices A and B are absolutely equivalent for all bounded sequences (Cooke[1], p. 133)

We may ask whether (AB) and (BA) are F-absolutely equivalent for all bounded sequences provided that the F-regular matrices A and B are F-absolutely equivalent for all bounded sequence. In fact we have more than we need (see Theorem 3.1 below)

In what follows we write

$$A \sim A'$$

to denote that the F-regular matrices A and A' are F-absolutely equivalent for all bounded sequences.

Theorem 3.1—If A, A', B, B' are F-regular matrices and if

$$A \sim A' \text{ and } B \sim B'$$

then

$$AB \sim A'B'.$$

PROOF : It is enough to prove that if A, A', C are F-regular and if $A \sim A'$, then

$$AC \sim A'C \tag{5}$$

and

$$CA \sim CA'. \quad ...(6)$$

For applying (5) with $C = B$ we get $AB \sim A'B$, applying (6) with A, A' replaced by B, B' and C replaced by A', we get $A'B \sim A'B'$, and the result follows.

To prove (5), let z be any bounded sequence. Since C satisfies (i) of Theorem 1.1, Cz is also bounded. Hence, by the definition of assertion that $A \sim A'$, it follows that

$$f\text{-lim } \{(A\,(Cz))_n - (A'\,(Cz))_n\} = 0.$$

By (4), this gives that

$$f\text{-lim } \{((AC)\,z)_n - ((A'\,C)\,z)_n\} = 0$$

and, since this holds for any bounded z, (5) holds.

For (6), let z be any bounded sequence. Then (again by definition)

$$f\text{-lim } \{(Az)_n - (A'\,z)_n\} = 0.$$

Since C is F-regular, it follows that

$$f\text{-lim } \{[C\,(Az - A'\,z)_n]\} = 0.$$

Using (4), we get that

$$f\text{-lim } \{[(CA)\,z]_n - [(AA')\,z]_n\} = 0$$

and the result follows.

4. F-ABSOLUTE TRANSLATIVE MATRICES

Let $z = (z_k)$ be a sequence. We define the sequence $w = (w_k)$ by $w_k = z_{k-1}$, (where we take $z_{-1} = 0$). As known, the convergence of (z_k) implies the convergence of (w_k) (to the same value) and conversely. A similar result also holds for almost convergent sequences. We may expect that, if z is A-summable to s then w is A-summable to s and conversely. But this is not necessarily the case (see Cooke[1], p.p. 113-19 and Powell[5], p. 42). To discuss the corresponding problem for the almost summaility we first need :

Definition 4.1—Let A be an F-regular matrix. We say that A is F-absolutely translative for bounded sequences if, for all bounded (z_n), $(z_n - z_{n-1})$ is almost A-summable to zero; that is to say that (u_n) is almost convergent to zero, where

$$u_n = \sum_{k=0}^{\infty} a_{nk} (z_k - z_{k-1}). \qquad \ldots(7)$$

While this is the natural definition, we can deduce an alternative from which is more convenient for applications. We have

$$\sum_{k=0}^{m} a_{nk} (z_k - z_{k-1}) = \sum_{k=0}^{m-1} (a_{nk} - a_{n,k+1}) z_k + a_{nm} z_m. \qquad \ldots(8)$$

If A is F-regular then it follows from (i) of Theorem 1.1 that, for fixed n, $a_{nm} \to 0$ as $m \to \infty$. Hence if (z_m) is bounded, then $a_{nm} z_m \to 0$. Thus letting $m \to \infty$ in (8), we see that (7) may be written as

$$u_n = \sum_{k=0}^{\infty} (a_{nk} - a_{n,k+1}) z_k \qquad \ldots(9)$$

Our final result is an analogue of Theorem 5.6. iv in Cooke[1] (p. 119).

Theorem 4.2—Let A be an F-regular matrix. Then A is always F-absolutely translative for all bounded sequences (z_k).

PROOF : If $| z_k | \leqslant M$ (for all k), then

$$\left| \frac{1}{q+1} \sum_{i=0}^{q} u_{n+i} \right| = \left| \sum_{k=0}^{\infty} \frac{1}{q+1} \sum_{i=0}^{q} (a_{n+i,k} - a_{n+i,k+1}) z_k \right|$$

$$\leqslant M \sum_{k=0}^{\infty} \frac{1}{q+1} \left| \sum_{i=0}^{q} (a_{n+i,k} - a_{n+i,k+1}) \right| \to 0$$

as $q \to \infty$ uniformly in n, by (iv) of Theorem 1.1, whence the result.

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