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ON α-QUASI CONVEX FUNCTIONS

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We define a new class of analytic functions $Q(\alpha)$ of α-Quasi convex functions as follows: Let $\alpha \geqslant 0$. $Q(\alpha)$ is the class of all functions $f(z)$ holomorphic in the open unit disk $U = \{z : |z| < 1\}$ with $f(0) = 0$, $f'(0) = 1$ and satisfying

$$\left| (1-\alpha) \arg \frac{zf'(z)}{g(z)} + \alpha \arg \frac{[zf'(z)]'}{g'(z)} \right| < \pi/2, \; z \in U$$

for some α-convex function (Mocanu sense) $g(z)$. $Q(0)$ is the class C of close-to-convex functions of Kaplan[4] and $Q(1)$ is the class Q of quasi-convex functions of Noor and Thomas[3]. Thus $Q(\alpha)$ unifies these two classes of functions. We prove in this paper that when $\alpha \geqslant 1$ all functions in $Q(\alpha)$ are close-to-convex (hence univalent) and the class $Q(\alpha)$ is invariant under certain integral operator.

1. INTRODUCTION

Let M_α denote the class of α-convex functions introduced by Mocanu[5]. This class M_α unifies the classes K- of convex univalent functions and S^* of starlike univalent functions. In fact $M_0 = S^*$ and $M_1 = K$. M_α gives a continuous passage from the class of convex univalent functions to the class of starlike univalent functions as α decreases from 1 to 0.

Let $P(\alpha)$ denote the class of all functions $f(z)$ holomorphic in U with $f(0) = 0$, $f'(0) = 1$ and satisfying for $\alpha \geqslant 0$

$$\int_{\theta_1}^{\theta_2} \text{Re} \left\{ (1-\alpha) \frac{zf'(z)}{f(z)} + \alpha \left(1 + \frac{zf''(z)}{f'(z)} \right) \right\} d\theta > -\pi$$

where

$$0 < \theta_1 < \theta_2 \leqslant \theta_1 + 2\pi, \; z = re^{i\theta}, \; r < 1.$$

This class was introduced and studied by Bharathi[1]. Also it was proved[1] that a function $f \in P(\alpha)$ if and only if $\exists$ a starlike univalent function $g(z)$ such that

$$\text{Re} \left\{ \frac{z^\alpha f'^\alpha(z) f^{1-\alpha}(z)}{g(z)} \right\} > 0 \quad \text{for } z \in U. \qquad ...(1)$$

Noor and Thomas[3] introduced the class Q of Quasi-convex functions as the class of all functions f holomorphic in U with $f(0) = 0$, $f'(0) = 1$ and satisfying

$$\operatorname{Re} \left\{ \frac{[zf'(z)]'}{g'(z)} \right\} > 0, \quad z \in U \qquad \ldots(2)$$

for some convex univalent function $g(z)$ with $g(0) = 0$, $g'(0) = 1$. Noor and Thomas[3] also established that

$$K \subset Q \subset C.$$

The purpose of this paper is to introduce a new class $Q(\alpha)$ of α-quasi convex functions. This new class will provide a continuous passage from Q to C as α decreases from 1 to 0 in an analogous manner M_α giving a continuous passage from K to S^*.

Throughout this paper wherever a non-integral power of analytic function is being used, it is assumed that a suitable analytic branch has to be considered.

2. Main Results

We now define the new class $Q(\alpha)$ of α-Quasi convex functions.

Definition 2.1—Let $\alpha \geqslant 0$. We denote by $Q(\alpha)$ the class of all functions $f(z)$ holomorphic in U with $f(0) = 0$, $f'(0) = 1$ and for some $g \in M_\alpha$ satisfying

$$\left| (1 - \alpha) \arg \frac{zf'(z)}{g(z)} + \alpha \arg \frac{[zf'(z)]'}{g'(z)} \right| < \pi/2, \; z \in U. \qquad \ldots(3)$$

Remark : It is clear that $Q(0)$ and $Q(1)$ are respectively the classes of close-to-convex functions and quasi-convex functions. Thus we have a continuous passage from Q to C as α decreases from 1 to 0. We also show that α-quasi-convex function is a close-to-convex function for $\alpha > 1$ and hence univalent in U.

We need the following theorem to establish our main results:

Theorem A—Let $\beta, \nu \in \mathbb{C}$, $h(z)$ be convex univalent in U with $h(0) = 1$ and $\operatorname{Re}(\beta h(z) + \nu) > 0$, $z \in U$ and let $q(z)$ be holomorphic in U with $q(0) = 1$ and $q(z) \prec h(z)$ ($q(z)$ is sub-ordinate to $h(z)$ in U). If $p(z) = 1 + p_1 z + \ldots$ is holomorphic in U then

$$p(z) + \frac{zf'(z)}{\beta q(z) + \nu} \prec h(z) \Rightarrow p(z) \prec h(z).$$

For proof of the result, See Padmanabhan and Parvatham[6].

Theorem 2.1—For $\alpha > 1$, $Q(\alpha) \subseteq Q(0) = C$.

Proof : Let $f \in Q(\alpha)$. Then $\exists$ a $g \in M_\alpha$ such that

$$\left| (1 - \alpha) \arg \frac{zf(z)}{g(z)} + \alpha \arg \frac{[zf'(z)]'}{g'(z)} \right| < \pi/2, \; z \in U$$

equivalently,

$$\operatorname{Re}\left\{\frac{[zf'(z)]^{1-\alpha}\,[zf'(z)]'^{\alpha}}{g^{1-\alpha}(z)\,g'^{\alpha}(z)}\right\} > 0,\ z \in U.$$

Thus

$$\frac{[zf'(z)]'\,[zf'(z)]^{(1-\alpha)/\alpha}}{g^{(1-\alpha)/\alpha}(z)\,g'(z)} = p^{1/\alpha}(z) \qquad ...(4)$$

where

$$|\arg p(z)| < \pi/2.$$

Now let

$$P(z) = \frac{zf'(z)}{g(z)}.$$

Then

$$\frac{[zf'(z)]'}{g'(z)} = P(z) + \frac{zP'(z)}{zg'(z)/g(z)}.$$

Substituting in (4) we get,

$$\left[P(z) + \frac{zP'(z)}{\frac{zg'(z)}{g(z)}}\right] P^{(1-\alpha)/\alpha}(z) = p^{1-\alpha}(\alpha).$$

Setting

$$P^{1/\alpha}(z) = p_1(z)$$

and

$$\frac{zg'(z)}{g(z)} = q(z)$$

we get

$$p_1(z) + \frac{zp_1'(z)}{q(z)} = p^{1/\alpha}(z).$$

Since $g \in M_\alpha \subseteq S^*$, $\operatorname{Re} q(z) > 0$ in U and $|\arg p^{1/\alpha}(z)| < \pi/2\alpha$, for $\alpha \geqslant 1$. $p^{1/\alpha}(z)$ lies in-side the convex set $|\arg \xi| < \pi/2\alpha$. Thus application of Theorem A with $\beta = 1$, $\gamma = 0$ and $h(z) = [(1-z)/(1+z)]^\alpha$ gives $|\arg p_1(z)| < \pi/2\alpha$ or $|\arg P(z)| < \pi/2\alpha$ which implies $f \in Q(0) = C$.

Remark : Since every function f in $Q(\alpha)$ is a close-to-convex function, $Q(\alpha)$ is a proper sub-class of the class S of normalized univalent function whenever $\alpha \geqslant 1$.

Theorem 2.2—$Q(\alpha) \subseteq Q(\beta)$ for $\alpha \geqslant 1$ and $0 \leqslant \beta \leqslant \alpha$.

PROOF : The case $\beta = 0$ has been established in Theorem 2.1, hence we assume $\beta > 0$. Let $f \in Q(\alpha)$. Then $\exists$ a $g \in M_\alpha$ such that for $z \in U$.

$$\left|(1-\alpha)\arg p(z) + \alpha \arg\left(p(z) + \frac{zp'(z)}{q(z)}\right)\right| < \pi/2$$

where

$$p(z) = \frac{zf'(z)}{g(z)}$$

and

$$q(z) = \frac{zg'(z)}{g(z)}.$$

Now

$$\left| (1-\beta) \arg p(z) + \beta \arg \left(p(z) + \frac{zp'(z)}{g(z)} \right) \right|$$

$$\leqslant \frac{\beta}{\alpha} \left| (1-\alpha) \arg p(z) + \alpha \arg \left(p(z) + \frac{zp'(z)}{q(z)} \right) \right|$$

$$+ \left(1 - \frac{\beta}{\alpha} \right) |\arg p(z)| \leqslant \frac{\beta}{\alpha} \frac{\pi}{2} + \left(1 - \frac{\beta}{\alpha} \right) \frac{\pi}{2} = \pi/2.$$

Thus $f \in Q(\beta)$ under the stated conditions of the theorem.

Theorem 2.3—$f \in Q(\alpha)$ if and only if $zf' \in P(\alpha)$.

PROOF : Since $f \in Q(\alpha)$ we have

$$\operatorname{Re} \left\{ \frac{z^{\alpha} (zf'(z))^{1-\alpha} [zf'(z)]^{\alpha}}{z^{\alpha} g^{1-\alpha}(z) \, g^{\alpha}(z)} \right\} > 0$$

for $z \in U$, where $g \in M_{\alpha}$. Equivalently, $\exists \, \phi \in S^*$ such that

$$Z^{\alpha} g^{1-\alpha}(z) \, g'^{\alpha}(z) = \phi(z)$$

and thus

$$\operatorname{Re} \left\{ \frac{z^{\alpha} [zf'(z)]^{1-\alpha} [zf'(z)]'^{\alpha}}{\phi(z)} \right\} > 0, \; z \in U.$$

From the representation theorem for the class $P(\alpha)$[1] we have $zf' \in P(\alpha)$. Converse follows immediately from the representation theorem for the class $P(\alpha)$.

Remark : When $\alpha = 1$, $Q(\alpha)$ and $P(\alpha)$ are noting but the classes Q and C respectively. Thus Theorem 1 of Noor and Thomas[3] is a particular case of Theorem 2.3. When $\alpha = 0$, $Q(0) = C$ and $P(0) = CS^*$ of Reade[7]. Thus Theorem 2.3 reduces to the well-known result namely $f \in C$ if and only if $zf' \in CS^*$.

Theorem 2.4—For α real, $\bigcap_{\alpha \geqslant 1} Q(\alpha) = K$.

PROOF : Let $f \in Q(\alpha)$, then $\exists$ a $g \in M_{\alpha}$ such that for $z \in U$

$$\left| \left(\frac{1}{\alpha} - 1 \right) \arg \frac{zf'(z)}{g(z)} + \arg \frac{[zf'(z)]'}{g'(z)} \right| < \frac{\pi}{2\pi}.$$

Allowing α to ∞ we get,

$$\left| \arg \frac{[zf'(z)]'}{g'(z)} - \arg \frac{zf'(z)}{g(z)} \right| = 0$$

that is

$$\left| \arg \frac{[zf'(z)]'}{zf'(z)} \frac{g(z)}{g'(z)} \right| = 0.$$

Therefore there exists a positive real number m such that

$$\frac{[zf'(z)]'}{f'(z)} \cdot \frac{g(z)}{zg'(z)} = m$$

which is the same as saying

$$1 + \frac{zf''(z)}{f'(z)} = m \frac{zg'(z)}{g(z)} .$$

Since $g \in M_\alpha \subseteq S^*$ and $m > 0$ we get f is a convex function. Hence from (2.2) it follows that $\bigcap_{\alpha \geqslant 1} Q(\alpha) \subseteq K$. Conversely, if f is a convex function then by taking $g(z) = f(z)$ we see that $f \in Q(\infty)$. Thus $\bigcap_{\alpha \geqslant 1} Q(\alpha) \supseteq K$. Hence Theorem 2.4 follows.

Remark 3 : By Theorems 2.3 and 2.4 we get

$$\bigcap_{\alpha \geqslant 1} P(\alpha) = S^*.$$

Theorem 2.5—Whenever $f \in Q(\alpha)$ and $\alpha \geqslant 1$, then $F(z)$ defined by

$$F(z) = \left\{ \frac{2/\alpha}{z^{2/\alpha}} \int_0^z t^{(2-\alpha)/\alpha} f'^{1/\alpha}(t)\, dt \right\}^\alpha \qquad \text{...(5)}$$

is also in $Q(\alpha)$.

PROOF : Since $f \in Q(\alpha)$ $\exists$ a $g \in M_\alpha$ such that

$$\mathrm{Re} \left\{ \frac{z^\alpha [zf'(z)]^{1-\alpha} [zf'(z)]^{\,\alpha}}{z^\alpha g^{1-\alpha} g^{\alpha}(z)} \right\} > 0 \quad \text{for } z \in U.$$

Let $G(z)$ be defined as,

$$G(z) = \left\{ \frac{1/\alpha}{z^{1/\alpha}} \int_0^z t^{(1-\alpha)/\alpha} g^{1/\alpha}(t)\, dt \right\}^\alpha .$$

Then

$$z^{1/\alpha} G^{1/\alpha}(z) = \frac{1}{\alpha} \int_0^z t^{(1-\alpha)/\alpha} g^{1/\alpha}(t)\, dt.$$

This on differentiation, with respect to z gives

$$\frac{1}{\alpha} z^{(1-\alpha)/\alpha} G^{1/\alpha}(z) + \frac{1}{\alpha} z^{1/\alpha} G^{(1-\alpha)/\alpha}(z) G'(z) = \frac{1}{\alpha} z^{(1-\alpha)/\alpha} g^{1/\alpha}(z).$$

Hence,

$$zG'(z) G^{(1-\alpha)/\alpha}(z) + G^{1/\alpha}(z) = g^{1/\alpha}(z)$$

again differentiating this with respect to z we get

$$\left(\frac{1}{\alpha} - 1\right) zG'^2(z) G^{(1-2\alpha)/\alpha}(z) + G'(z) G^{(1-\alpha)/\alpha}(z) + zG^{(1-\alpha)/\alpha}(z) G''(z)$$
$$+ \frac{1}{\alpha} G^{(1-\alpha)/\alpha}(z) G'(z)$$
$$= \frac{1}{\alpha} g^{(1-\alpha)/\alpha}(z) g'(z).$$

That is

$$(1-\alpha) zG^{(1-2\alpha)/\alpha}(z) G'^2(z) + \alpha zG^{(1-\alpha)/\alpha}(z) G''(z)$$
$$+ (1+\alpha) G^{(1-\alpha)/\alpha}(z) G'(z)$$
$$= g^{(1-\alpha)/\alpha}(z) g'(z).$$

Therefore,

$$G^{(1-\alpha)/\alpha}(z) G'(z)\left[(1+\alpha) + \frac{\alpha zG''(z)}{G'(z)} + (1-\alpha)\frac{zG'(z)}{G(z)}\right]$$
$$= g^{(1-\alpha)/\alpha}(z) g'(z) \qquad \text{...(6)}$$

putting

$$p(z) = (1-\alpha)\frac{zG'(z)}{G(z)} + \left(1 + \frac{zG''(z)}{G(z)}\right)$$

(6) becomes

$$G^{(1-\alpha)/\alpha}(z) G'(z)[p(z) + 1] = g^{(1-\alpha)/\alpha}(z) g'(z) \qquad \text{...(7)}$$

taking logarithamic differentiation of (7) we get

$$p(z) + \frac{\alpha zp'(z)}{p(z)+1} = (1-\alpha)\frac{zg'(z)}{g(z)} + \alpha\left(1 + \frac{zg''(z)}{g'(z)}\right).$$

As $g \in M_\alpha$ we get

$$\text{Re}\left(p(z) + \frac{\alpha zp'(z)}{p(z)+1}\right) > 0$$

equivalently

$$p(z) + \frac{\alpha zp'(z)}{p(z)+1} \prec \frac{1+z}{1-z}.$$

Using a result due to Eenignburg *et al.*[2] we get

$$p(z) \prec \frac{1+z}{1-z} \quad \text{or} \quad \operatorname{Re} p(z) > 0.$$

Thus $G \in M_\alpha$ whenever $g \in M_\alpha$.

Now let

$$P(z) = \frac{[zF'(z)]^{1-\alpha}[zF'(z)]'\alpha}{G^{1-\alpha}(z)\, G^{\alpha}(z)}$$

equivalently,

$$[zF'(z)]^{(1-\alpha)/\alpha}[zF'(z)]' = P^{1/\alpha}(z)\, G^{(1-\alpha)/\alpha}(z)\, G'(z). \quad \ldots(8)$$

Also we have,

$$zF''(z) + F'(z) = \frac{P^{1/\alpha}(z)\, G^{(1-\alpha)/\alpha}(z)\, G'(z)}{[zF'(z)]^{(1/\alpha)-1}}. \quad \ldots(9)$$

From the representation (5) for $F'(z)$ we have

$$z^{1/\alpha} F'(z)^{1/\alpha} = \frac{2}{\alpha}\int_0^z t^{(2-\alpha)/\alpha} f'^{1/\alpha}(t)\, dt$$

which on differentiation with respect to z yields,

$$z^{1/\alpha} \cdot \frac{1}{\alpha} F'(z)^{(1-\alpha)/\alpha} F''(z) + F'^{1/\alpha}(z)\frac{2}{\alpha} z^{(2-\alpha)/\alpha}$$

$$= \frac{2}{\alpha} z^{(2-\alpha)/\alpha} f'^{1/\alpha}(z).$$

Hence,

$$F^{(1-\alpha)/\alpha}(z)\,[zF''(z) + 2F'(z)] = 2f'^{1/\alpha}(z).$$

Using (9) we get,

$$F^{(1-\alpha)/\alpha}(z)\left\{\frac{P^{1/\alpha}(z)\, G^{(1-\alpha)\,\alpha}(z)\, G'(z)}{[zF(z)]^{(1-\alpha)/1}} + F'(z)\right\} = 2f'^{(1-\alpha)/\alpha}(z)$$

or

$$\frac{P^{1/\alpha}(z)\, G^{(1-\alpha)/\alpha}(z)\, G'(z)}{z^{(1-\alpha)/\alpha}} + F'^{1/\alpha}(z) = 2f'^{1/\alpha}(z).$$

Therefore,

$$zP'^{1/\alpha}(z)\, G^{(1-\alpha)/\alpha}(z)\, G'(z) + (zF'(z))^{1/\alpha} = 2\,[zf'(z)]^{1/\alpha}. \quad \ldots(10)$$

Differentiating (10) with respect to z and simplifying using (8) and (6) we get

$$\frac{1}{\alpha}\, P^{1/\alpha} g^{(1-\alpha)/\alpha} g' + \frac{z}{\alpha}\, P^{(1-\alpha)/\alpha} P' G^{(1-\alpha)/\alpha} G' = \frac{2}{\alpha}\, (zf')^{(1-\alpha)/\alpha} (zf')'.$$

Therfore

$$P^{1/\alpha} + \frac{zP^{(1-\alpha)/\alpha} P'}{\left(\frac{g^{(1-\alpha)/\alpha} g'}{G^{(1-\alpha)/\alpha} G'}\right)} = 2\, \frac{(zf')^{(1-\alpha)/\alpha} (zf')'}{g^{(1-\alpha)/\alpha} g'}\,. \quad ..(11)$$

Setting

$$P_1(z) = P^{1/\alpha}(z); \; Q(z) = \frac{g^{(1-\alpha)/\alpha}(z)\, g'(z)}{G^{(1-\alpha)/\alpha}(z)\, G'(z)}$$

(11) becomes

$$P_1(z) + \frac{\alpha z P_1'(z)}{Q(z)} = \frac{2\,(zf')^{(1-\alpha)/\alpha} (zf')'}{g^{(1-\alpha)/\alpha} g'}\,.$$

From (6) we get

$$Q(z) = \alpha\left(1 + \frac{zG''(z)}{G'(z)}\right) + (1-\alpha)\, \frac{zG(z)}{G(z)} + 1$$

since $G \in M_\alpha$ we find $\operatorname{Re} Q(z) > 0$. Also as $f \in Q(\alpha)$ we have

$$\left|\arg\left(P_1(z) + \frac{\alpha z P_1'(z)}{Q(z)}\right)\right| \leqslant \frac{\pi}{2\alpha}$$

an application of Theorem A for $\alpha \geqslant 1$ yields:

$$|\arg P_1(z)| \leqslant \frac{\pi}{2\alpha}\;; \text{ or } \; |\arg P(z)| \leqslant \frac{\pi}{2}$$

which gives $F \in Q(\alpha)$ thereby establishing the theorem.

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