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EDGE CRACK IN ORTHOTROPIC ELASTIC HALF-PLANE

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Problems of determining the distribution of stress and displacement in an orthotropic elastic half-plane containing an edge crack normal to the free-surface when (I) the shape of the crack is prescribed and (II) the cracks are opened by given normal pressure, have been considered. Numerical results, for various loading functions, of stress intensity factor and crack energy have been presented for problem II taking into account the values of constants for boron-epoxy composite.

1. INTRODUCTION

The problems of determining stress and displacement fields in an isotropic elastic half-plane containing an edge crack have been considered by Sneddon and Das[10], Krapkhov[5], Srivastav and Narain[12], Irwin[3], Koiter[4] and many others. Details of references can be had in Sih[9], Sneddon and Lowengrub[11]. The authors used different techniques like integral transform method, complex variable method, alternating method, Wiener-Hopf technique etc. As such crack problems in orthotropic medium are important and useful from technological point of view. Dhaliwal[2], Satapathy and Parhi[8], Kushwaha[6] and Das and Behera[1] have considered Griffith cracks in orthotropic medium. In this paper we have dealt with two edge crack problems normal to the boundary in an orthotropic elastic half plane :

(I) having prescribed crack shape and the other, (II) having been opened by prescribed normal pressures.

The displacements and stresses in a two dimensional orthotropic elastic medium are expressed in terms of two potential functions[2,8], which are harmonic in two different planes both being different from the physical plane considered. Using integral transform technique closed form solution is obtained for problem I whereas the solution of problem II have been reduced in solving Fredholm integral equation of second kind which is suitable for numerical computation.

In problem I, expression for pressure necessary to produce the Griffith crack of prescribed shape have been obtained. Also in problem II, expressions for quantities of physical interest e.g. shape of the crack, stress intensity factor and crack energy have been obtained.

Numerical computations for problem II have been done for boron-epoxy composite material for linearly loaded, partially constant loaded, point loaded and constant loaded cracks.

2. The Basic Equations

For an orthotropic medium, we choose the Cartesian co-ordinate axes to be coincident with the principal material axes. For the case of plane strain following Satapathy and Parhi[8] the displacement and stress components are expressed in terms of two potential functions φ_1 and φ_2 as

$$u_x = \frac{\partial \varphi_1}{\partial x} + \frac{\partial \varphi_2}{\partial x} \quad \ldots(2.1)$$

$$u_y = \frac{\lambda_1}{\delta_1} \frac{\partial \varphi_1}{\partial y_1} + \frac{\lambda_2}{\delta_2} \frac{\partial \varphi_2}{\partial y_2} \quad \ldots(2.2)$$

$$\sigma_{xx}/A_{66} = \frac{1+\lambda_1}{\delta_1^2} \frac{\partial^2 \varphi_1}{\partial x^2} + \frac{1+\lambda_2}{\delta_2^2} \frac{\partial^2 \varphi_2}{\partial x^2} \quad \ldots(2.3)$$

$$\sigma_{yy}/A_{66} = (1+\lambda_1) \frac{\partial^2 \varphi_1}{\partial y_1^2} + (1+\lambda_2) \frac{\partial^2 \varphi_2}{\partial y_2^2} \quad \ldots(2.4)$$

$$\sigma_{xy}/A_{66} = \frac{1+\lambda_1}{\delta_1} \frac{\partial^2 \varphi_1}{\partial x\, \partial y_1} + \frac{1+\lambda_2}{\delta_2} \frac{\partial^2 \varphi_2}{\partial x\, \partial y_2} \quad \ldots(2.5)$$

φ_1 and φ_2 satisfy the differential equation

$$\frac{\partial^2 \varphi_i}{\partial x^2} + \frac{\partial^2 \varphi_i}{\partial y_i^2} = 0, \; (i = 1, 2) \quad \ldots(2.6)$$

where

$$y_i = y/\delta_i \; (i = 1, 2) \quad \ldots(2.7)$$

and λ_1, λ_2 and δ_1^2, δ_2^2 are the roots of the quadratic equations

$$\lambda^2 A_{66} \, (A_{12} + A_{66}) + \lambda \, [(A_{12} + A_{66})^2 + A_{66}^2 - A_{11} A_{22}] + A_{66} (A_{12} + A_{66}) = 0 \quad \ldots(2.8)$$

and

$$\delta^4 A_{11} A_{66} + \delta^2 [(A_{12} + A_{66})^2 - A_{66}^2 - A_{11} A_{12}] + A_{22} A_{66} = 0 \quad \ldots(2.9)$$

respectively. A_{ij} are anisotropic constants of the orthotropic material.

3. Formulation of the Problems

We consider the semi-infinite region $x \geqslant 0$ and take the half plane to be orthotropic containing a Griffith edge crack $0 \leqslant x \leqslant a$ on $y = 0$ normal to the free surface

$x = 0$. Due to symmetry with respect to the x-axis the problems are reduced to quarter plane $(x > 0,\ y > 0)$ problems.

Problem I (Displacement prescribed)

The problem is to determine the components of displacements and stresses at any point and to determine the pressure $p(x)$ necessary to produce the crack of prescribed shape $w(x)$. The boundary conditions of the problem are

$$u_y(x, 0) = w(x),\ 0 \leqslant x \leqslant a \qquad \ldots(3.1)$$

$$u_y(x, 0) = 0,\ x > a \qquad \ldots(3.2)$$

$$\sigma_{xy}(x, 0) = 0,\ x \geqslant 0 \qquad \ldots(3.3)$$

$$\sigma_{xx}(0, y) = 0,\ y \geqslant 0 \qquad \ldots(3.4)$$

and

$$\sigma_{xy}(0, y) = 0,\ y \geqslant 0. \qquad \ldots(3.5)$$

Problem II (Stress prescribed)

The crack is opened by equal and opposite prescribed pressures $p_0 f(x)$ normal to the faces of it. The problem is to determine the components of displacements and stresses at any point and to determine the shape of the crack, stress intensity factor at the crack tip and the crack energy. The boundary conditions of the problem are

$$\sigma_{yy}(x, 0) = -p_0 f(x),\ 0 \leqslant x \leqslant a \qquad \ldots(3.6)$$

together with (3.2) to (3.5).

4. Solution of the Problems

We take solutions of eqns. (2.6) in the form

$$\varphi_1(x, y) = \int_0^\infty [\alpha^{-1} B_1(\alpha)\, e^{-\alpha \delta_1 x} \cos \alpha y + \alpha^{-2} C_1(\alpha)\, e^{-\alpha y_1} \cos \alpha x]\, d\alpha \qquad \ldots(4.1)$$

and

$$\varphi_2(x, y) = \int_0^\infty [\alpha^{-1} B_2(\alpha)\, e^{-\alpha \delta_2 x} \cos \alpha y + \alpha^{-2} C_2(\alpha)\, e^{-\alpha y_2} \cos \alpha x]\, d\alpha \qquad \ldots(4.2)$$

where $B_1(\alpha)$, $B_2(\alpha)$, $C_1(\alpha)$ and $C_2(\alpha)$ are unknown functions to be determined. Boundary conditions (3.3) and (3.5) will be satisfied, if we take

$$\frac{1 + \lambda_1}{\delta_1} C_1(\alpha) = -\frac{1 + \lambda_2}{\delta_2} C_2(\alpha) \qquad \ldots(4.3)$$

and

$$B_1(\alpha) = -\frac{1+\lambda_2}{1+\lambda_1}\frac{\delta_2}{\delta_1}B_2(\alpha). \qquad ...(4.4)$$

From boundary conditions (3.2) and (3.4), using (4.3) and (4.4), we obtain

$$\int_0^\infty \alpha^{-1} C_2(\alpha)\cos\alpha x\, d\alpha = 0, \quad \text{for all } x > a \qquad ...(4.5)$$

$$\frac{(\delta_1-\delta_2)}{\delta_1}\int_0^\infty B_2(\alpha)\,\alpha\cos\alpha y\, d\alpha$$

$$+\frac{1}{\delta_2}\int_0^\infty\left[\frac{e^{-\alpha y_1}}{\delta_1}-\frac{e^{-\alpha y_2}}{\delta_2}\right]C_2(\alpha)\,d\alpha = 0, y \geqslant 0. \qquad ...(4.6)$$

Problem I

Boundary condition (3.1) with the help of (4.3) gives

$$\int_0^\infty C_2(\alpha)\,\alpha^{-1}\cos\alpha x\, d\alpha = \theta(x),\ 0 \leqslant x \leqslant a \qquad ...(4.7)$$

where

$$\theta(x) = \frac{(1+\lambda_1)\,\delta_2}{(\lambda_1-\lambda_2)}w(x). \qquad ...(4.8)$$

Now eqns. (4.5) – (4.7) will determine the unknown functions $B_2(\alpha)$ and $C_2(\alpha)$. Taking

$$C_2(\alpha) = \alpha\int_0^a t\,\varphi(t)\,J_0(\alpha t)\,dt$$

$$B_2(\alpha) = \int_0^\infty t\,\psi(t)\,J_0(\alpha t)\,dt \qquad ...(4.9)$$

eqn. (4.5) is identically satisfied and eqn. (4.7)

leads to (c. f. Magnus and Oberhettinger[7])

$$\int_x^a \frac{t\,\varphi(t)\,dt}{(t^2-x^2)^{1/2}} = \theta(x), 0 \leqslant x \leqslant a$$

which has the solution

$$\varphi(t) = \frac{2}{\pi}\left[\frac{\theta(a)}{(a^2-t^2)^{1/2}} - \int_t^a \frac{\theta'(u)\,du}{(u^2-t^2)^{1/2}}\right], 0 \leqslant t \leqslant a. \qquad ...(4.10)$$

Equation (4.6) with the help of (4.9) and (4.10) gives

$$\int_0^y \frac{t\,\psi(t)\,dt}{(y^2-t^2)^{1/2}} = \frac{2\delta_1}{\pi\delta_2(\delta_1-\delta_2)} \int_0^a u\left[\frac{1}{\left(y_1^2+u^2\right)^{1/2}}\right.$$

$$\left.-\frac{1}{\left(y_2^2+u^2\right)^{1/2}}\right]\times\left[\frac{\theta(a)}{(a^2-u^2)^{1/2}}-\int_u^a \frac{\theta'(\beta)\,d\beta}{(\beta^2-u^2)^{1/2}}\right]du$$

which has the solution

$$t\,\psi(t) = \left(\frac{2}{\pi}\right)^2 \frac{\delta_1}{\delta_2(\delta_1-\delta_2)} \int_0^a u\left[\frac{\theta(a)}{(a^2-u^2)^{1/2}} - \int_u^a \frac{\theta'(\beta)d\beta}{(\beta^2-u^2)^{1/2}}\right]$$

$$\times\left[\frac{d}{dt}\int_0^t \frac{y}{(t^2-y^2)^{1/2}}\left(\frac{1}{\left(y_1^2+u^2\right)^{1/2}}\right.\right.$$

$$\left.\left.-\frac{1}{(y_2^2+u^2)^{1/2}}\right)dy\right]du.$$

Simplifying we get,

$$\psi(t) = \frac{4\delta_1 t}{\pi^2\,\delta_2\,(\delta_1-\delta_2)} \int_0^a \left[\frac{1}{t^2+u^2\,\delta_2^2} - \frac{1}{t^2+u^2\,\delta_1^2}\right]$$

$$\times\left[\frac{\theta(a)}{(a^2-u^2)^{1/2}} - \int_u^a \frac{\theta'(\beta)\,d\beta}{(\beta^2-u^2)^{1/2}}\right]du,\ t>0. \qquad \ldots(4.11)$$

Equation (4.10) and (4.11) give $\psi(t)$ and $\phi(t)$ in closed form. So $B_2(\alpha)$ and $C_2(\alpha)$ are known from (4.9). Then (4.3) and (4.4) give $C_1(\alpha)$ and $B_1(\alpha)$ respectively. Therefore the displacements and stresses can be determined from (2.1) – (2.5).

Expression for pressure $p(x)$ necessary to produce the Griffith crack of prescribed shape is obtained from (2.4) with the help of (4.1) – (4.4) and (4.9) as

$$p(x) = A_{66}(1+\lambda_2)\left[\delta_2\,x\int_0^\infty\left\{\frac{\delta_1^2}{(t^2+\delta_1^2\,x^2)^{3/2}} - \frac{\delta_2^2}{(t^2+\delta_2^2\,x^2)^{3/2}}\right\}\right.$$

$$\left.\times\, t\,\psi(t)\,dt + \left(1-\frac{\delta_1}{\delta_2}\right)\frac{d}{dx}\int_0^x \frac{t\,\varphi(t)\,dt}{(x^2-t^2)^{1/2}}\right],\ 0\leqslant x\leqslant a.$$

Problem II

Boundary condition (3.6) with the help of (4.3) and (4.4) gives

$$\delta_2 \int_0^\infty \alpha\, B_2(\alpha)\,[\delta_1\, e^{-\alpha \delta_1 x} - \delta_2\, e^{-\alpha \delta_2 x}]\, d\alpha + \left(1 - \frac{\delta_1}{\delta_2}\right)$$

$$\times \int_0^\infty C_2(\alpha) \cos \alpha x \, d\alpha = \frac{-p_0 f(x)}{A_{66}(1 + \lambda_2)}, \; 0 \leqslant x \leqslant a. \quad \ldots(4.12)$$

In this case, (4.5), (4.6) and (4.12) will determine the unknown functions $B_2(\alpha)$ and $C_2(\alpha)$. Taking the same integral representations given in (4.9) for $B_2\,\alpha$ and $C_2(\alpha)$ as in problem I, eqn. (4.5) is identically satisfied and eqn. (4.6) leads to

$$\int_0^y \frac{t\,\psi(t)\,dt}{(y^2 - t^2)^{1/2}} = \frac{\delta_1}{\delta_2(\delta_1 - \delta_2)} \int_0^a u\,\varphi(u)$$

$$\times \left[\frac{1}{(y_1^2 + u^1)^{1/2}} - \frac{1}{(y_2^2 + u^2)^{1/2}}\right] du. \quad \ldots(4.13)$$

Inverting (4.13) we get

$$\psi(t) = \int_0^a K_2(t, u)\; \varphi(u)\, du, \; t \geqslant 0 \quad \ldots(4.14)$$

where

$$K_2(t, u) = \frac{2\delta_1\, t}{\pi\, \delta_2(\delta_1 - \delta_2)} \left[\frac{1}{t^2 + u^2\,\delta_2^2} - \frac{1}{t^2 + u^2\,\delta_1^2}\right]. \quad \ldots(4.15)$$

Equation (4.12) under (4.9) becomes

$$\int_0^x \frac{t\,\varphi(t)\,dt}{(x^2 - t^2)^{1/2}} = \frac{\delta_2}{(\delta_1 - \delta_2)} \left[\delta_2 \int_0^\infty u\,\psi(u) \left\{\frac{1}{(\delta_2^2\, x^2 + u^2)^{1/2}}\right.\right.$$

$$\left.\left. - \frac{1}{(\delta_1^2\, x^2 + u^2)^{1/2}}\right\} du + \frac{p_0\, F(x)}{A_{66}(1 + \lambda_2)}\right] \quad \ldots(4.16)$$

where

$$F(x) = \int_0^x f(x)\, dx. \quad \ldots(4.17)$$

Inverting (4.16) we get

$$\varphi(t) = \int_0^\infty K_3(t, u)\,\psi(u)\; du + F_1(t), \; 0 \leqslant t < a \quad \ldots(4.18)$$

where

$$K_3(t, u) = \frac{2}{\pi} \frac{\delta_2^2 \, t}{(\delta_1 - \delta_2)} \left[\frac{\delta_1^2}{t^2 \delta_1^2 + u^2} - \frac{\delta_2^2}{t^2 \delta_2^2 + u^2} \right] \quad \text{...(4.19)}$$

and

$$F_1(t) = \frac{2}{\pi} \frac{\delta_2 \, p_0}{(\delta_1 - \delta_2) A_{66} (1 + \lambda_2)} \int_0^t \frac{f(x) \, dx}{(t^2 - x^2)^{1/2}}. \quad \text{...(4.20)}$$

The solution of the problem, therefore, reduces to the solution of the pair of simultaneous integral eqns. (4.14) and (4.18).

Eliminating $\psi(t)$ from (4.14) and (4.18) we get

$$\varphi(t) - \int_0^a \varphi(v) \, L(t, v) \, dv = F_1(t), \; 0 \leqslant t \leqslant a \quad \text{...(4.21)}$$

where

$$L(t, v) = \frac{4 \, \delta_1 \delta_2 \, t}{\pi^2 (\delta_1 - \delta_2)^2} \left[\frac{\delta_1^2}{(t^2 \delta_1^2 - v^2 \delta_2^2)} \log \left(\frac{t \, \delta_1}{v \, \delta_2} \right) + \frac{\delta_2^2}{(t^2 \delta_2^2 - v^2 \delta_1^2)} \log \left(\frac{t \, \delta_2}{v \, \delta_1} \right) - \frac{2}{(t^2 - v^2)} \times \log \left(\frac{t}{v} \right) \right]. \quad \text{...(4.22)}$$

This kernel $L(t, v)$ in eqn. (4.22) appears to have singularities at points $t = v \, \delta_2/\delta_1$, $v \, \delta_1/\delta_2$ and $t = v$. But it can be easily shown by L' Hospital's rule that $L(t, v)$ tends to finite limits when t tends to these three points. Accordingly, eqn. (4.21) is suitable for its numerical solution. Thus quantities of physical interest of the problem can be determined. The stress intensity factor K_I is given by

$$K_I = \lim_{x \to a+} [2(x - a)]^{1/2} \sigma_{yy}(x, 0)$$

$$= \frac{A_{66} (1 + \lambda_2)(\delta_1 - \delta_2)}{\delta_2} a^{1/2} \varphi(a). \quad \text{...(4.23)}$$

Crack energy W is given by

$$W = - \int_0^a \sigma_{yy}(x, 0) \, u_y(x, 0) \, dx$$

$$= \frac{p_0 (\lambda_1 - \lambda_2)}{(1 + \lambda_1) \, \delta_2} \int_0^a t \, \varphi(t) \left\{ \int_0^t \frac{f(x) \, dx}{(t^2 - x^2)^{1/2}} \right\} dt. \quad \text{...(4.24)}$$

The shape of the crack is obtained from (2.2) using (2.8), (4.3) and (4.9) as

$$u_y(x, 0) = \frac{(1-\lambda_2)}{\delta^2} \int_x^a \frac{t\ \varphi(t)\,dt}{(t^2 - x^2)^{1/2}}, \quad 0 \leqslant x \leqslant a. \qquad \ldots(4.25)$$

Taking the values of the elastic constants as

$$A_{66} = 1.13 \times 10^6 \text{ psi}, \lambda_1 = 17.430, \lambda_2 = 0.057.$$

$\delta_1^2 = 2.862$ and $\delta_2^2 = 0\ 047$ for Boron-Epoxy composite (c.f. Das and Behera[1], K_I, W and $\varphi(t)$ in (4.21) for different t, for unit crack length have been computed for various loading functions $f(x)$. The results obtained are given in Table I and presented graphically in Fig. 1 respectively.

TABLE I

Loading function $f(x)$	*Stress intensity factor* (K_I)	*Crack energy* (W)
1	1.06893 p_0	0.23598 p_0^2
$H(x - \frac{1}{2})$	1.04894 p_0	0.33632 p_0^2
x	0.66221 p_0	0.09149 p_0^2
$\delta(x - 1/2)$	0.68220 p_0	0.60762 p_0^2

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