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PROPAGATION OF ALFVÉN WAVES IN A REAL MAGNETOHYDRODYNAMIC FLUID

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The system of Alfvén waves in a real MHD fluid is discussed in detail. The equation of state and thermodynamic quantities of the system have been obtained with the help of the formulae of Bose-Einstein statistics and electrodynamics. It has been found that if we ignore the viscous forces from the equation of state of Alfvén waves in a real MHD fluid, the equation reduces to the equation of state of the system of Alfvén waves in an ideal MHD fluid.

1. INTRODUCTION

In this paper we consider Alfvén waves as a system. The energy of this system is quantized by the boundary conditions imposed on the walls of the box containing the real fluid. We calculate the density of states and then the free energy of the system with the help of the formulae of electrodynamics. This leads us to find the equation of state of the system.

2. DEFINITION

We consider the system of Alfvén waves in a real MHD fluid and assume that quantum of energy of an Alfvén wave exists having the value $\hbar w$ ($\hbar$ being Dirac h and w the angular frequency). This quantum is named as Alfvén on[1].

3. FORMULATION OF THE PROBLEM

We take the system of Alfvén waves in a cubic box having each side a and volume V and assume that the system is in thermal equilibrium with the box having perfectly reflecting and ideally conducting walls. The wave function vanishes on all sides of the box. Like photons, Alfvénons are subject to quantum statistics and especially to Bose-Einstein statistics. Moreover, they disobey pauli exclusion principle. We shall find different thermodynamic quantities of the system with the help of the formulae of electromagnetic waves.

4. SOLUTION OF THE PROBLEM

We write the basic equations of magnetohydrodynamics for a real MHD fluid as follows[2] :

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0 \qquad \ldots(1)$$

$$\rho \frac{d\vec{v}}{dt} = -\nabla p + \frac{1}{4\pi}\{(\nabla \times \vec{B}) \times \vec{B}\} + (\eta + 1/3\,\mu)\,\nabla(\nabla \cdot \vec{v}) + \mu\nabla^2 \vec{v} \qquad \ldots(2)$$

$$\frac{\partial \vec{B}}{\partial t} = \nabla \times (\vec{v} \times \vec{B}) + \nu m\, \nabla^2 \vec{B} \qquad \ldots(3)$$

$$\nabla \cdot \vec{B} = 0 \qquad \ldots(4)$$

$$\rho T \frac{ds}{dt} = Q \qquad \ldots(5)$$

$$p = p(\rho, T) \qquad \ldots(6)$$

where $\rho = \rho(\vec{r}, t)$ is the mass density of fluid, $\vec{v} = \vec{v}(\vec{r}, t)$ its velocity, p the pressure, $\vec{r}$ the displacement of the fluid, $\vec{B}$ the magnetic field, μ and η are the first and second coefficients of viscosity, νm the magnetic viscosity, T the absolute temperature, S the entropy and Q the energy density dissipated per unit time. Moreover, the last two terms in eqn. (2) represent the viscous forces. We assume that the waves propagating in the fluid are the plane harmonic waves having small amplitudes. With the help of this assumption the eqns. (1)–(5) can be linearized. Hence, we write

$$\vec{v} = \vec{v}_0 + \vec{v}_1 = \vec{v}_1$$

$$\vec{B} = \vec{B}_0 + \vec{B}_1$$

$$p = p_0 + p_1 \qquad \ldots(7)$$

$$\rho = \rho_0 + \rho_1$$

where $\vec{v}_0$, $\vec{B}_0$, p_0 and ρ_0 are constants and correspond to the uniform state of the fluid. We take $\vec{v}_0 = 0$ at equilibrium state. Moreover, $\vec{v}_1$, $\vec{B}_1$, p_1 and ρ_1 are small perturbations in the quantities $\vec{v}_0$, $\vec{B}_0$, p_0 and ρ_0 respectively. The values of these perturbations and their derivatives always remain less than the constant quantities and therefore, we neglect all but the linear terms in $\vec{v}_1$, $\vec{B}_1$, p_1 and ρ_1. Hence, eqns (1)–(5) are linearized as :

$$\frac{\partial \rho_1}{\partial t} + \rho_0 (\nabla \cdot \vec{v}_1) = 0 \qquad \ldots(8)$$

$$\rho_0 \frac{\partial \vec{v}_1}{\partial t} = -\nabla p_1 + \frac{1}{4\pi}\{(\nabla \times \vec{B}_1) \times \vec{B}_0\} + (\eta + 1/3\,\mu)\,\nabla(\nabla \cdot \vec{v}_1)$$

$$+ \mu \nabla^2 \vec{v}_1 \qquad ...(9)$$

$$\frac{\partial \vec{B}_1}{\partial t} = \nabla \times (\vec{v}_1 \times \vec{B}_0) + \nu m \nabla^2 \vec{B}_1 \qquad ...(10)$$

$$\nabla \cdot \vec{B}_1 = 0 \qquad ...(11)$$

and

$$\rho_0 T \frac{ds}{dt} = Q. \qquad ...(12)$$

This system of homogeneous, linear partial differential equations governs the behaviour of the perturbation in space and time. The assumption that the waves propagating in the fluid are plane harmonic waves, helps us to simplify eqns. (8)–(11). We introduce the plane wave solution :

$$A = A_1 e^{i(\vec{r} \, . \, \vec{k} \cdot wt)} \qquad ...(13)$$

where A is any fluctuating quantity, A_1 its amplitude, w the angular frequency, $i = \sqrt{-1}$ and $\vec{K}$ the wave vector. Thus the simplified form of the eqns. (8)–(11) will be :

$$\rho_1 w - \rho_0 (\vec{K} \cdot \vec{v}_1) = 0 \qquad ...(14)$$

$$\rho_0 w \vec{v}_1 = p_1 \vec{K} + \frac{1}{4\pi} \{(\vec{B}_0 \cdot \vec{B}_1) \vec{K} - (\vec{K} \cdot \vec{B}_0) \vec{B}_1\} - i \mu K^2 \vec{v}_1$$

$$- i (\eta + 1/3 \mu) (\vec{K} \cdot \vec{v}_1) \vec{K} \qquad ...(15)$$

$$w \vec{B}_1 = (\vec{K} \cdot \vec{v}_1) \vec{B}_0 - (\vec{K} \cdot \vec{B}_0) \vec{v}_1 - i \nu m K^2 \vec{B}_1 \qquad ...(16)$$

$$\vec{K} \cdot \vec{B}_1 = 0. \qquad ...(17)$$

The expression for the speed of sound gives

$$p_1 = \frac{\rho_1}{\gamma} C_s^2 \qquad ...(18)$$

where $\gamma = Cp/Cv$; Cp and Cv are specific heats at constant pressure and volume respectively. With the help of eqns. (14)–(18), we may obtain the following relation between the kinetic and magnetic energies for all types of modes.

$$1/2 \, \rho_0 v_1^2 = \frac{\left\{ C_s^2 \Big/ \gamma - i (\eta_1 + 1/3 \, \nu) w \right\} \rho_1^2 w / \rho_0}{2 (w + i \nu K^2)}$$

$$+ \frac{(w + i \nu m K^2)}{8\pi (w + i \nu K^2)} B_1^2 \qquad ...(19)$$

where $\eta_1 = \eta/\rho_0$ and $\nu = \mu/\rho_0$ is the kinematic viscosity. Also substitution of $\vec{B}_1$ from eqn. (16) into eqn. (15) and then simplification with the help of the eqns. (14) and (18) gives the dispersion relation for general modes, given below :

$$[\rho_0 w + i \mu K^2 - \frac{(\vec{K} \cdot \vec{B}_0)^2}{4\pi (w + i \nu m K^2)}] \vec{v}_1 = \Big[\Big\{ \frac{\rho_0 C_s^2}{\gamma w} + \frac{B_0^2}{4\pi (w + i \nu m K^2)}$$

$$- i (\eta + 1/3 \mu) \Big\} \vec{K} - \frac{(\vec{K} \cdot \vec{B}_0) \vec{B}_0}{4\pi (w + i \nu m K^2} \Big] (\vec{K} \cdot \vec{v}_1)$$

$$- \frac{(\vec{K} \cdot \vec{B}_0) (\vec{B}_0 \cdot \vec{v}_1)}{4\pi (w + i \nu m K^2)} \vec{K}. \qquad ...(20)$$

Now corresponding to different modes of oscillatons this equation gives us different dispersion relations. Moreover, calculating $\vec{B}_1$ from eqn. (10) and substituting into eqn. (9) and then simplifying we can show that the velocity of Alfvén waves will be :

$$\vec{C}_A = \frac{\vec{B}_0}{\sqrt{4\pi \rho_0}}. \qquad ...(21)$$

The condition that the fluid velocity $\vec{v}_1$ must be perpendicular to the unperturbed magnetic field $\vec{B}_0$ and the wave vector $\vec{K}$ for the propagation of Alfvén waves gives :

$$\vec{K} \cdot \vec{v}_1 = 0 = \vec{B}_0 \cdot \vec{v}_1. \qquad ...(22)$$

We take $\vec{B}_0$ along z-axis and λ as the angle between $\vec{B}_0$ and $\vec{K}$.

Thus,

$$\vec{K} \cdot \vec{B} = K B_0 \cos \lambda. \qquad ...(23)$$

Thus eqns. (14), (18) and (22) jointly give

$$\rho_1 = 0 = p_1. \qquad ...(24)$$

We can simplify the eqns. (19)–(20) with the help of the eqns. (23)–(24). Thus we get

$$1/2 \rho_0 v_1^2 = \frac{(w + i \nu m K^2) B_1^2}{8\pi (w + i \nu K^2)} \qquad ...(25)$$

and

$$\nu m \nu K^4 + \Big[C_A^2 \cos^2 \lambda \, i (\nu m + \nu) w \Big] K^2 - w^2 = 0. \qquad ...(26)$$

Equation (25) is the energy equation and (26) the dispersion relation for the Alfvén waves propagating in a real MHD fluid. Thus the compressibility of the fluid

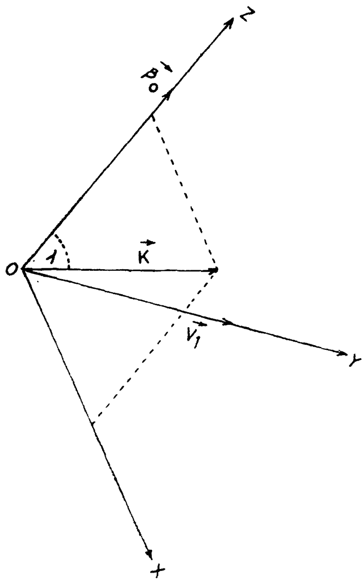

FIG. 1. Direction of the magnetic field $\mathbf{B}_0$, the wave vector $\mathbf{K}$ and the fluid velocity $\mathbf{v}_1$.

does not play any role in Alfvén waves, density and pressure perturbations do not accompany these waves, energy flow is always along the magnetic lines of forces, geometrical spreading of the energy does not take place and the medium remains unchanged thermodynamically. Moreover, there is equipartition between hydrodynamic and magnetic energies. Equation (26) gives the following relations for phase and group velocities respectively

$$u = w/K = \left\{ C_A^2 \cos^2 \lambda - i\,(\nu m + \nu)\, w \right\}^{1/2} \qquad \ldots(27)$$

and

$$\frac{\partial w}{\partial \vec{\mathbf{K}}} = \frac{2\, C_A^2 \cos^2 \lambda - i\,(\nu m + \nu)\, w}{2\, C_A^2 \cos^2 \lambda + i\,(\nu m + \nu)\, K\, C_A \cos \lambda}\, \vec{\mathbf{C}}_A. \qquad \ldots(28)$$

Due to the boundary conditions imposed by us on the walls of the box, the oblique propagation of MHD waves is impossible. The wave propagation will be either along or across the magnetic field. Here we shall consider only the propagation of Alfvén waves along the magnetic field. Thus $\lambda = 0$ and eqn. (27) can be reduced to

$$K = \frac{w}{C_A} + i\, \frac{L w^2}{2\, C_A^3}$$

where $L = \nu m + \nu$

$$dK = \left[\frac{1}{C_A} + i \frac{Lw}{C_A^3} \right] dw. \qquad \ldots(29)$$

Since the propagation of Alfvén waves is along one side of the cubic box only; therefore, one dimensional formula for density of states will be[3]

$$dg\,(K) = \frac{a}{\pi}\, dK.$$

Considering all types of polarizations of Alfvén waves, this will become :

$$dg\,(K) = \frac{2\,V^{1/3}}{\pi}\, dK \qquad \ldots(30)$$

where $a = V^{1/3}$.

Substitution of dK from eqn. (29) into eqn. (30) gives

$$dg\,(w) = \frac{2\,V^{1/3}}{\pi} \left[\frac{1}{C_A} + i\, \frac{Lw}{C_A^3} \right] dw. \qquad \ldots(31)$$

Now we calculate the thermodynamic quantities of the system. Using the formula for free energy ξ of the system as given by Guggenheim[4] :

$$\xi = \theta \sum_k g_k \ln\,(1 - \exp\, -E_k/\theta).$$

It can be written in integral form as

$$\xi = \theta \int_0^\infty \ln\,(1 - \exp\,(-\hbar w/\theta))\, dg \qquad \ldots(32)$$

where $E_k = \hbar w$ is the energy of kth Alfvénon and $\theta = kT$ (k being Boltzmann's constant and T the absolute temperature). Substituting equation (31) into equation (32), we get

$$\xi = \frac{2V^{1/3}\,\theta}{\pi\, C_A} \int_0^\infty \ln\,(1 - \exp\,(-\hbar w/\theta))\, dw$$

$$+ \frac{2iLV^{1/3}\,\theta}{\pi\, C_A^3} \int_0^\infty w \cdot \ln\,(i - \exp\,(\hbar w/\theta))\, dw.$$

Let

$$\hbar w/\theta = x$$

$$\therefore \xi = \frac{2\,V^{1/3}\,\theta^2}{\pi\,C_A\,\hbar}\int_0^\infty \ln\,(1 - e^{-x})\,dx$$

$$+\ \frac{2\,iLV^{1/3}\,\theta^3}{\pi\,\hbar^2\,C_A^3}\int_0^\infty x\cdot\ln\,(1 - e^{-x})\,dx. \quad \text{...(33)}$$

The formula for power series of logarithmic functions is given below :

$$\ln\,(1 - e^{-x}) = -\sum_{n=1}^{\infty}\frac{e^{-x_n}}{n}.$$

We simplify the expression (33) as follows :

$$\xi = \frac{-2\,V^{1/3}\,\theta^2}{\pi\,\hbar\,C_A}\int_0^\infty\sum_{n=1}^{\infty}\frac{e^{-nx}}{n}\,dx - \frac{2\,iLV^{1/3}\,\theta^3}{\pi\,\hbar^2\,C_A^3}\int_0^\infty\sum_{n=1}^{\infty}\frac{x\cdot e^{-nx}}{n}\,dx.$$

Let

$$nx = y$$

$$\therefore \xi = \frac{-2V^{1/3}\,\theta^2}{\pi\,\hbar\,C_A}\sum_{n=1}^{\infty}\left(\frac{1}{n^2}\int_0^\infty e^{-y}\,dy\right)$$

$$-\ \frac{2\,iLV^{1/3}\,\theta^3}{\pi\,\hbar^2\,C_A^3}\left(\sum_{n=1}^{\infty}\frac{1}{n^3}\int_0^\infty ye^{-y}\,dy\right). \quad \text{...(34)}$$

The integrals in eqn. (34) can be calculated as :

$$\int_0^\infty e^{-y}\,dy = 1 = \int_0^\infty y\,e^{-y}\,dy$$

$$\therefore \xi = \frac{2\,V^{1/3}\,\theta^2}{\pi\,\hbar\,C_A}\sum_{n=1}^{\infty}\frac{1}{n^2} - i\,\frac{2\,LV^{1/3}\,\theta^3}{\pi\,\hbar^2\,C_A^3}\sum_{n=1}^{\infty}\frac{1}{n^3}. \quad \text{...(35)}$$

Now

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \pi^2/6$$

and

$$\sum_{n=1}^{\infty} \frac{1}{n^3} = 1.202$$

$$\therefore \xi = \frac{-\pi V^{1/3} \theta^2}{3\hbar C_A} - i \frac{(2.4) L V^{1/3} \theta^3}{\pi \hbar^2 C_A^3} \qquad \text{...(36)}$$

The entropy of the system will be

$$S = -\left(\frac{\partial \xi}{\partial \theta}\right)_V$$

$$S = \frac{2\pi V^{1/3} \theta}{3 \hbar C_A} + i \frac{(7.2) L V^{1/3} \theta^2}{\pi \hbar^2 C_A^3} . \qquad \text{...(37)}$$

The pressure p will be

$$p = -\left(\frac{\partial \xi}{\partial V}\right)\theta$$

$$p = \frac{\pi V^{-2/3} \theta^2}{9 \hbar C_A} + i \frac{(0.8) L V^{-2/3} \theta^3}{\pi \hbar^2 C_A^3} \qquad \text{...(38)}$$

$$p V^{2/3} = \frac{\pi \theta^2}{9 \hbar C_A} + i \frac{(0.8) L \theta^3}{\pi \hbar^2 C_A^3} \qquad \text{...(39)}$$

It is the equation of state for Alfvén waves propagating in a real MHD fluid. Thermodynamic potential $M = \xi + pV$

$$M = -\frac{2\pi V^{1/3} \theta^2}{9 \hbar C_A} - i \frac{(1.6) L V^{1/3} \theta^3}{\pi \hbar^2 C_A^3} . \qquad \text{...(40)}$$

The mean energy $\bar{E} = \xi + \theta S$

$$\therefore \bar{E} = \frac{\pi V^{1/3} \theta^2}{\pi \hbar C_A} + i \frac{(4.8) L V^{1/3} \theta^3}{\pi \hbar^2 C_A^3} . \qquad \text{...(41)}$$

The enthalpy $y = \bar{E} + pV$.

So

$$Y = \frac{4\,\pi\,V^{1/3}\,\theta^2}{9\,\hbar\,C_A} + i\,\frac{(5.6)\,LV^{1/3}\,\theta^3}{\pi\,\hbar^2\,C_A^3}\,. \qquad \ldots(42)$$

The specific heat at constant volume $C_V = \left[\frac{\partial \bar{E}}{\partial \theta}\right]_V$

$$C_V = \frac{2\pi\,V^{1/3}\,\theta}{3\hbar\,C_A} + i\,\frac{(14.4)\,LV^{1/3}\,\theta^2}{\pi\,\hbar^2\,C_A^3}\,. \qquad \ldots(43)$$

The specific heat at constant pressure C_p is given by

$$C_p - C_V = -\frac{\theta\left[\frac{\partial p}{\partial \theta}\right]_V^2}{\left[\frac{\partial p}{\partial V}\right]_\theta}\,.$$

Therefore,

$$C_p = \frac{\neg\pi\,V^{1/3}\,\theta}{3\,\hbar^2\,C_A} + i\,\frac{(21.6)\,LV^{1/3}\,\theta^2}{\pi\,\hbar^2\,C_A^3}\,. \qquad \ldots(44)$$

The ratio of specific heats $\gamma = C_p/C_V$ can be calculated as :

$$\gamma = 1.54.$$

5. Conclusion

From the equation of state of the system of Alfvén waves in a real MHD fluid, we may conclude that the equation consists of real and imaginary parts. The real parts are non-viscous while the imaginary part appearing on the right hand side of the equation is due to viscous forces. If we ignore the viscous forces, eqn. (39) will be reduced to the equation of state of the system of Alfvén waves in an ideal MHD fluid[1].

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