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ON STRONGLY NBD-FINITE FAMILIES

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The purpose of this paper is to give the properties of strongly nbd-finite families of Topological Spaces.

INTRODUCTION

The concept of strongly nbd-finite family of a topological space has been introduced in Thangavelu[6] in order to study the piecewise definition of certain maps such as irresolute[1], semicontinuous[4] etc. The purpose of the present paper is to obtain several properties of strongly nbd-finite families. In fact Theorem 2.10 is revised as per the suggestions given by the referee. The concepts of semiopen set, semiclosed set, α-set (Alpha set) and Semiclosure are discussed respectively in Levine[4], Crossley[1], Njastad[5] and Crossley[1].

1. PRELIMINARIES

By X we always mean a topological space on which no separation axioms are assumed unless explicitly stated. If A and B are subsets, $A - B$ denotes the difference of B in A. Listed below are definitions and results that will be utilized in this paper.

Definition 1.1[6]—A family $\{A_m : m \in M\}$ of Subsets of X is said to be strongly nbd-finite if for each x in X, there is an open set V containing x, satisfying one of the following conditions :

(a) $V \cap A_m = \phi$ for every $m \in M$

(b) There is a non-empty finite subset N of M such that

(i) $V \cap A_m \neq \phi$ for every $m \in N$

(ii) $V \cap A_m \subset A_k$ for every m, k with $m \in N, K \in N$ and

(iii) $V \cap A_m = \phi$ for every $m \in M - N$.

Theorem 1.2[6]—If $\{A_m : m \in M\}$ is strongly nbd-finite then $\{\text{Scl}(A_m) : m \in M\}$ is also strongly nbd-finite in X where Scl (A) is the semiclosure of A, the largest semiclosed set that contains A.

Theorem 1.3[6]—The union of Semiclosed Sets of a strongly nbd-finite family is semiclosed.

Definition 1.4[2]—A space X is semi-regular if and only if for each semiclosed set A and $x \in X - A$, there exist disjoint Semiopen Sets U and V such that $x \in U$ and $A \subset V$.

Theorem 1.5[2]—X is semi-regular if and only if for each x and for each semi-open set A containing x, there is a semi open set B with $x \in B \subset \text{scl}(B) \subset A$.

2. Properties

From the Definition 1.1 it is obvious that a covering $\{A_m : m \in M\}$ of X is strongly nbd-finite if and only if the condition (1.1) (b) holds. Further a subfamily of a strongly nbd-finite family is also strongly nbd-finite and if $\{A_m : m \in M\}$ is strongly nbd-finite in X then $\{A \cap A_m : m \in M\}$ is also strongly nbd-finite in the subspace A of X.

The proof of the next Theorem is analogous to that of Theorem 1.2.

Theorem 2.1—If $\{A_m : m \in M\}$ is strdngly nbd-finite then $\{\text{Cl}(A_m) : m \in M\}$ is also strongly nbd-finite where Cl (A) is the closure of A.

Theorem 2.2—Let $\{A_m : m \in M\}$ be any family of sets in X. If the Union of semiclosures of A_m, $m \in M$ is semiclosed then it is equal to the semiclosure of the Union of A_m, $m \in M$.

Proof : Follows from the properties of Semiclosure.

Corollary 2.3—If $\{A_m : m \in M\}$ is any strongly nbd-finite family of sets then $\cup \{\text{scl}(A_m) : m \in M\} = \text{scl}(\cup \{A_m : m \in M\})$.

Proof : Follows from Theorems 1.2, 1.3 and 2.2.

Theorem 2.4—If $\{A_i : i \in I\}$ and $\{B_j : j \in J\}$ are strongly nbd-finite families of X then $\{A_i \cap B_j : (i, j) \in I \times J\}$ is also strongly nbd-finite.

Proof : Fix x in X. There are open sets V_1 and V_2 both containing x and satisfying the following conditions :

Condition 2.5—(a) $V_1 \cap A_i = \phi$ for every i in I:

or

(b) There is a non-empty finite subset N_1 of I such that (i) $V_1 \cap A_i \neq \phi$ for every i in N_1, (ii) $V_1 \cap A_i \subset A_k$ for every i, k with $i \in N_1$, $K \in N_1$ and (iii) $V_1 \cap A_i = \phi$ for every $i \in I - N_1$.

Condition 2.6—(a) $V_2 \cap B_j = \phi$ for every $j \in J$

or

(b) There is a non-empty finite subset N_2 of J such that (i) $V_2 \cap B_j \neq \phi$ for every $j \in N_2$, (ii) $V_2 \cap B_j \subset B_l$ for every j, l with $j \in N_2$, $l \in N_2$ and (iii) $V_2 \cap B_j = \phi$ for every $j \in J - N_2$.

Let $V = V_1 \cap V_2$. Then V is open and contains x. Now $I \times J = L_1 \cup L_2 \cup L_3 \cup L_4$ where $L_1 = N_1 \times N_2$, $L_2 = N_1 \times (J - N_2)$, $L_3 = (I - N_1) \times N_2$ and $L_4 = (I-N_1) \times (J - N_2)$. It is easy to verify that $V \cap A_i \cap B_j = \phi$ for every ordered pair (i, j) in $L_2 \cup L_3 \cup L_4$. If this is also true for every pair (i, j) in L_1 then the Theorem is proved. Suppose not, let $N = \{(i, j) \in L_1 : V \cap A_i \cap B_j \neq \phi\}$. Clearly N is finite and non-empty. Let (i, j) and $(k, 1)$ be in N. Then using Condition 2.5(b) and Condition 2.6 (b) we have $V \cap A_i \cap Bj = (V_1 \cap B_i) \cap (V_2 \cap B_j) \subset A_k \cap B_1$. This proves the Theorem.

If a covering of X has a nbd-finite refinement then it also has a precise nbd-finite refinement. The next Theorem is an analog of this.

Theorem 2.7—If the covering $\{A_i : i \in I\}$ of X has a strongly nbd-finite refinement $\{B_m : m \in M\}$ then it also has a precise strongly nbd-finite refinement $\{C_k : k \in I\}$. Further more if each B_m is open (resp. Semiopen, resp. α-set, resp. closed and resp. Semiclosed) then C_k can be chosen to be open (resp. semiopen. resp. α-set, resp, closed and resp. semiclosed).

PROOF : For each m in M fix an $i \in I$ such that $B_m \subset A_i$. Having done this, define a map $F : M \to I$ defined by $F(m) = i$ if $B_m \subset A_i$. For each i in I take $C_i = \cup \{B_m : F(m) = i\}$. Clearly $C_i \subset A_i$ for each i in I and $\{C_k : k \in I\}$ is a covering of X and hence a precise refinement of $\{A_i : i \in I\}$.

Claim : $\{C_k : k \in I\}$ is strongly nbd-finite.

Fix x in X. Since $\{B_m : m \in M\}$ is a strongly nbd-finite covering of X, Choose V as in 1.1 Satisfying 1.1 (b) where A is replaced by B. Take $L = F(N) = \{F(m) : m \in N\}$. Clearly L is finite and non-empty. If $i \in L$ then $F(m) = i$ for some $m \in N$ so that, using 1.1 (b) (i), V intersects B_m which implies V intersects C_i. Let $i \in I-L$. Now $N \subset F^{-1}(F(N)) = F^{-1}(L)$. Taking complement, we get $M - N \supset F^{-1}(I-L)$ so that $F^{-1}(i) \subset M - N$ which implies, by using 1.1 (b) (iii), V does not intersect B_m for every m with $F(m) = i$. This proves that $V \cap C_i = \phi$. Let $k \in L$. Now $V \cap C_k = V \cap (\cup \{B_m : F_{no} : F(m) = k\}) = \cup \{V \cap B_m : F(m) = k\}$ which is contained in B_n for every $n \in N$ (by 1.1 (b) (ii)). This implies that $V \cap C_k \subset C_{F(n)}$ for every $n \in N$. This completes the proof of the claim.

If each B_m is open (resp. semiopen, resp. α-set) then C_k is open (resp. semiopen, resp. α-set). If each B_m is closed, since every strongly nbd-finite family is nbd-finite and since union of closed sets of a nbd-finite family is closed, C_k is closed for each k in I. If each B_m is semiclosed then, using, Theorem 1.3, C_k is semiclosed for each k in I. This proves the Theorem.

The next Theorem is the generalization of Theorem 1.5 of Dugundji[3] (p. 162).

Theorem 2.8—Let $\{E_i : i \in I\}$ be any family of Sets in a space X and let $\{B_m : m \in M\}$ be a strongly nbd-finite Closed (resp. Semiclosed) pairwise disjoint covering of X. Assume that each B_m intersects at most finitely sets E_i. Then there is a strongly nbd-finite family $\{C(E_i) : i \in I\}$ such that $E_i \subset C(E_i)$ and $C(E_i)$ is clopen (resp. semiclopen), for each i in I.

PROOF : For each $i \in I$, define $C(E_i) = X - \cup \{B_m : B_m \cap E_i = \phi\}$. Since $\{B_m : m \in M\}$ is pairwise disjoint $C(E_i) = \cup \{B_m : B_m \cap E_i \neq \phi\}$. If each B_m is closed, since every strongly nbd-finite family is nbd-finite, each $C(E_i)$ is both open and closed. If each B_m is semiclosed, by using Theorem 1.3 and Theorem 2.2, each $C(E_i)$ is both semiopen and semiclosed. Further, for each i in I, it is easy to see that $C(E_i)$ intersects B_m if and only if E_i intersects B_m. Also $E_i \subset C(E_i)$ for each i in I.

Claim—$\{C(E_i) : i \in I\}$ is strongly nbd-finite.

Fix x in X. Since $\{B_m : m \in M\}$ is a strongly nbd-finite covering of X. Choose V and N as in 1.1 satisfying Definition 1.1 (b) where where A is replaced by B. Since $\{B_m : m \in M\}$ is pairwise disjoint, N is singleton namely $\{m_0\}$. Now using Definition 1.1 (b), $V \cap B_{m_0} \neq \phi$ and $V \cap B_m = \phi$ for all m different from m_0. Since $\{B_m : m \in M\}$ is a covering, $V \subset B_{m_0}$. Since each B_m intersects at most finitely many E_i, Choose a finite subset N_0 of I such that $B_{m_0} \cap E_i \neq \phi$ for every $i \in N_0$ and $B_{m_0} \cap E_i = \phi$ for every $i \in I - N_0$.

If $i \in I - N_0$ then $B_{m_0} \cap E_i = \phi$ which implies $C(E_i) \cap B_{m_0} = \phi$ so that $V \cap C(E_i) = \phi$. Let L denote the set of all i in N_0 such that $V \cap C(E_i) \neq \phi$. If L is empty then V does not intersect $C(E_i)$ for each i in I. Let L be non-empty. As N_0 is finite, L is finite. If i, j are in L then $V \cap C(E_i) = V \cap (\cup \{B_m : B_m \cap E_i \neq \phi\})$ which equals $V \cap B_{m_0} = V \subset B_{m_0}$.

Since $B_{m_0} \cap C(E_0) \neq \phi$, $B_{m_0} \subset C(E_j)$ so that $V \cap C(E_i) \subset C(E_j)$. This completes the proof.

Theorem 2 9—If X is semi-regular (resp. regular) and if each semiopen (resp. open) covering has a strongly nbd-finite refinement then each semiopen (resp. open) covering has a semiclosed (resp. closed) strongly nbd-finite refinement.

PROOF : We prove for semi-regular X and the proof for regular X is similar.

Let $\{A_i : i \in I\}$ be a semiopen covering of X. By Theorem 1.5, for each x in X and for each A_i with $x \in A_i$, there is a semiopen set B_i with $x \in B_i \subset \mathrm{Scl}(B_i) \subset A_i$. Now $\{B_i : i \in I\}$ is a semiopen covering for X. Therefore it has a strongly nbd-finite refinement $\{C_k : k \in K\}$. Then $\{\mathrm{Scl}(C_k): k \in K\}$ is a semiclosed strongly nbd-finite refinment of $\{A_i : i \in I\}$. This completes the proof.

Theorem 2.10—If each open covering of X has an open refinement that can be decomposed into an at most countable collection of strongly nbd-finite families of open sets, then each open covering has a strongly nbd-finite refinement.

PROOF : Let $\{U_i : i \in I\}$ be an open covering of X. Then it has an open refinement $\{V_{n,m} : (n, m) \in Z^+ X M\}$ where Z^+ is the set of positive integers and for each fixed $n_0 \in Z^+$, the family $\{V_{n_0,m} : m \in M\}$ is strongly nbd-finite. For each positive integer $n \in Z^+$ let W_n denote the Union of sets $V_{n,m}$ where m varies over the index set M. Then $\{W_n : n \in Z^+\}$ is an open covering for X. Take $A_1 = W_1$ and $A_n = W_n - \bigcup_{K<n} W_k$ for $n \geqslant 2$. If $y \in X$, let $n(y)$ be the first k such that $Y \in W_k$ so that $Y \in A_{n(y)}$. Therefore $\{A_n : n \in Z^+\}$ is a refinement of $\{W_k : k \in Z^+\}$. Now for each x in X, $W_{n(x)} \cap A_{n(x)} \neq \phi$ and $W_{n(x)} \cap A_k = \phi$ if $k \neq n(x)$. Thus $\{A_n : n \in Z^+\}$ is strongly nbd-finite.

Claim—$\{A_n \cap V_{n,m} : (n, m) \in Z^+ X M\}$ is strongly nbd-finite.

Fix x in X. Then $W_{n(x)} \cap A_k = \phi$ for every $k \neq n(x)$. Now $\{V_{n(x),m} - m \in M\}$ is strongly nbd-finite. Therefore there is an open set G containing x such that either $G \cap V_{n(x),m} = \phi$ for every m in M or there is a non-empty finite subset Δ of M such that (i) $G \cap V_{n(x),m} \neq \phi$ for every m in Δ, (ii) $G \cap V_{n(x),m} \subset V_{n(x),k}$ for every m, k with $m \in \Delta$, $k \in \Delta$ and (iii) $G \cap V_{n(x),m} = \phi$ for every m in $M - \Delta$. Take $W = W_{n(x)} \cap G$. Let

$$N = \{(n(x), m) : W \cap A_{n(x)} \cap V_{n(x),m} \neq \phi,\ m \in \Delta\}.$$

If $N = \phi$ the claim is proved. So, let $N \neq \phi$. Clearly N is finite. Let $(n(x), m)$ and $(n(x), k)$ be in N. Then $W \cap A_{n(x)} \cap V_{n(x),m} \subset G \cap A_{n(x)} \cap V_{n(x),m}$

$$\subset A_{n(x)} \cap V_{n(x),k}.$$

This completes the proof of the claim and hence the Theorem.

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