Indian J. pure appl. Math., **10(1) : 1-5, January 1979.**

AN ANISOTROPIC INCOHERENT FLUID COSMOLOGICAL MODEL IN GENERAL RELATIVITY

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(*Received* 25 *January* 1978)

A cosmological model of cylindrical symmetry with two degrees of freedom representing incoherent matter distribution has been derived. Various physical and geometrical properties of the model have been discussed.

1. INTRODUCTION

In recent years there has been a lot of interest in cosmological models which are non-isotropic and non-homogeneous. A plane symmetric cosmological model has been constructed by Singh and Singh (1968). Further work in this line has been done by Singh and Abdussattar (1973). In this paper, a cosmological model of cylindrical symmetry with two degrees of freedom representing incoherent matter distribution has been derived. The model represents an expanding universe in which the lines of flow of matter are geodetic, expanding, shearing but non-rotating. The density decreases with time. The space-time is of Petrov type I non-degenerate.

2. DERIVATION OF THE LINE ELEMENT

We consider the cylindrically symmetric metric with two degrees of freedom in the form

$$ds^2 = A^2(dx^2 - dt^2) + C^2dz^2 + (B^2 + D^2)\,dy^2 + 2CD\,dy\,dz \qquad ...(2.1)$$

where A, B, C and D are functions of t alone. The energy momentum tensor for incoherent matter distribution is given by

$$T_i^j = \epsilon V_i V^j \qquad ...(2.2)$$

together with

$$g_{ij}V^iV^j = -1 \qquad ...(2.3)$$

where $\epsilon > 0$ is the density and V^i is the flow vector satisfying (2.3). We assume the coordinates to be comoving so that

$$V^1 = V^2 = V^3 = 0 \text{ and } V^4 = \frac{1}{A}.$$

The field equations

$$R_i^j - \frac{1}{2} R \delta_i^j = -8\pi T_i^j \text{ (with } C = G = 1 \text{ and } \Lambda = 0) \quad \text{...(2.4)}$$

for the line-element (2.1) are as follows :

$$\frac{A_4 B_4}{AB} + \frac{A_4 C_4}{AC} - \frac{B_4 C_4}{BC} - \frac{B_{44}}{B} - \frac{C_{44}}{C} - \frac{1}{4}\left[\frac{CD_4 - DC_4}{BC}\right]^2 = 0 \quad \text{...(2.5)}$$

$$\frac{A_{44}}{A} + \frac{B_{44}}{B} - \frac{A_4^2}{A^2} + \frac{3}{4}\left[\frac{CD_4 - DC_4}{BC}\right]^2 = 0 \quad \text{...(2.6)}$$

$$\frac{A_{44}}{A} + \frac{C_{44}}{C} - \frac{A_4^2}{A^2} - \frac{1}{4}\left[\frac{CD_4 - DC_4}{BC}\right]^2 = 0 \quad \text{...(2.7)}$$

$$\frac{1}{A^2}\left[\frac{A_4 B_4}{AB} + \frac{B_4 C_4}{BC} + \frac{A_4 C_4}{AC}\right] - \frac{1}{4}\left[\frac{CD_4 - DC_4}{ABC}\right]^2 = 8\pi\epsilon \quad \text{...(2.8)}$$

$$\left[\frac{D}{C}\right]_{44} - \left[\frac{D}{C}\right]_4 \left[\frac{B_4}{B} - \frac{3C_4}{C}\right] = 0. \quad \text{...(2.9)}$$

The suffix 4 after the symbols A, B, C and D denotes ordinary differentiation with respect to t. From eqn. (2.9), we have

$$CD_4 - DC_4 = K\frac{B}{C} \quad \text{...(2.10)}$$

K being a constant of integration. From eqns. (2.6), (2.7) and (2.10), we obtain

$$\frac{C_{44}}{C} - \frac{B_{44}}{B} = \frac{K^2}{C^4}. \quad \text{...(2.11)}$$

Also from eqns. (2.5), (2.7) and (2.10), we get

$$\left[BC\left(\frac{A_4}{A}\right)\right]_4 = \frac{1}{2}[BC]_{44} \quad \text{...(2.12)}$$

which on integration gives

$$\frac{A_4}{A} = \frac{1}{2}\frac{(BC)_4}{BC} + \frac{L}{BC} \quad \text{...(2.13)}$$

L being a constant of integration. From eqns. (2.5), (2.10) and (2.13), we obtain

$$\left[\frac{B_4}{B}\right]_4 + \frac{1}{2}\left[\frac{B_4}{B}\right]^2 - \frac{L(BC)_4}{(BC)^2} = \frac{1}{2}\left[\frac{C_4}{C}\right]^2 - \frac{C_{44}}{C} - \frac{1}{4}\frac{K^2}{C^4}. \quad \text{...(2.14)}$$

Equations (2.11) and (2.14) lead to the following differential equations :

$$\left[\frac{B_4}{B}\right]_4 - \frac{2L(BC)_4}{(BC)^2} = -3\left[\frac{C_4}{C}\right]_4 - 2\left[\frac{C_4}{C}\right]^2 + \frac{1}{2}\frac{K^2}{C^4} \quad \text{...(2.15)}$$

and

$$\left[\frac{B_4}{B}\right]_4 + \frac{2L(BC)_4}{(BC)^2} = 4\left[\frac{C_4}{C}\right]_4 + 3\left[\frac{C_4}{C}\right]^2 - \frac{3}{2}\frac{K^2}{C^4}. \qquad ...(2.16)$$

Using substitutions $B = \exp\left(\frac{U + v}{2}\right)$ and $C = \exp\left(\frac{U - v}{2}\right)$ eqns. (2.15) and (2.16) lead to

$$v_{44} + U_4 v_4 = -K^2 \exp[2(v - U)] \qquad ...(2.17)$$

which on integration gives

$$v_4 \exp(U) = [M - K^2 \exp(2v)]^{1/2} \qquad ...(2.18)$$

M being a constant of integration. Also from eqns. (2.16) and (2.18), we obtain the following differential equation

$$\chi_{44} + \tfrac{4}{3}(\chi^{-3})_4 + \frac{M}{16}\chi^{-7} = 0 \qquad ...(2.19)$$

where $\chi = \exp(\frac{1}{4}U)$. For simplicity we take $L = 0$. Equation (2.19) then leads to

$$\chi_4 = \sqrt{N + \frac{M}{48\chi^6}} \qquad ...(2.20)$$

where N is a constant of integration. From eqns. (2.18) and (2.20) we get

$$\exp(-v) = \frac{K}{\sqrt{M}} \cosh\left[\log\left(\alpha \tan\frac{\theta}{2}\right)^{-4/\sqrt{3}}\right] \qquad ...(2.21)$$

where $\tan\theta = 4\chi^3\sqrt{\frac{3N}{M}}$ and α is a constant of integration.

Hence

$$B^2 = \frac{\sqrt{M}}{K}\chi^4 \operatorname{sech}\left[\log\left(\alpha \tan\frac{\theta}{2}\right)^{-4/\sqrt{3}}\right] \qquad ...(2.22)$$

and

$$C^2 = \frac{K}{\sqrt{M}}\chi^4 \cosh\left[\log\left(\alpha \tan\frac{\theta}{2}\right)^{-4/\sqrt{3}}\right]. \qquad ...(2.23)$$

Also eqn. (2.13) on integration gives

$$A = l\chi^2 \qquad ...(2.24)$$

l being a constant of integration. From eqns. (2.10), (2.18) and (2.21), we obtain

$$D = C\left[n - \frac{\sqrt{M}}{K}\tan h\left\{\log\left(\alpha \tan\frac{\theta}{2}\right)^{-4/\sqrt{3}}\right\}\right] \qquad ...(2.25)$$

where n is a constant of integration.

By a suitable transformation of coordinates, the metric of this model can be put into the form

$$ds^2 = (9a^2T^2 - 1)^{2/3}\,[dX^2 + \cosh P dZ^2 + (2nq \cosh P - 2 \sinh P)\, dY\, dZ$$
$$+ \{(q^2n^2 + 1)\cosh P - 2nq \sinh P\}\, dY^2] - dT^2 \qquad \ldots(2.26)$$

where a and q are arbitrary positive constants and

$$P = \log\left[\alpha^2\left(\frac{3aT - 1}{3aT + 1}\right)\right]^{-2/\sqrt{3}}.$$

3. Some Physical Features

The density for the space-time (2.26) is given by

$$8\pi\epsilon = \frac{12a^2}{(9a^2T^2 - 1)}\,. \qquad \ldots(3.1)$$

The reality condition $\epsilon > 0$ requires

$$9a^2T^2 > 1. \qquad \ldots(3.2)$$

The expression for expansion θ, rotation ω and shear σ_{ij} calculated for the flow vector V^i are given by

$$\theta = \frac{18a^2T}{(9a^2T^2 - 1)}, \qquad \ldots(3.3)$$

$$\omega = 0 \qquad \ldots(3.4)$$

and

$$\sigma_{22} = -\frac{2\sqrt{3}a \sinh P}{(9a^2T^2 - 1)^{1/3}} \qquad \ldots(3.5)$$

$$\sigma_{33} = \frac{2\sqrt{3}a}{(9a^2T^2 - 1)^{1/3}}\,[2nq \cosh P - (1 + n^2q^2)\sinh P] \qquad \ldots(3.6)$$

$$\sigma_{23} = \frac{2\sqrt{3}a}{(9a^2T^2 - 1)^{1/3}}\,[\cosh P - nq \sinh P] \qquad \ldots(3.7)$$

the other components of the shear tensor σ_{ij} being zero. Thus it is observed that the model represents an expanding universe in which the lines of flow of matter are geodetic, expanding, shearing but non-rotating.

The non-vanishing physical components of the conformal curvature tensor are given by

$$C_{(3434)} = -C_{(1212)} = \frac{4a^2}{(9a^2T^2 - 1)^2}\,[3\sqrt{3}aT \tanh P - 1] \qquad \ldots(3.8)$$

$$C_{(1313)} = -C_{(2424)} = \frac{4a^2}{(9a^2T^2 - 1)^2}\,[3\sqrt{3}\,aT \tanh P + 1] \qquad \ldots(3.9)$$

$$C_{(2323)} = -C_{(1414)} = -\frac{8a^2}{(9a^2T^2 - 1)^2} \qquad \ldots(3.10)$$

$$C_{(1213)} = C_{(2434)} = \frac{12\sqrt{3}\,a^3T \operatorname{sech} P}{(9a^2T^2 - 1)^2}. \qquad \ldots(3.11)$$

It follows that the space-time is of Petrov type I non-degenerate.

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