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CURVATURE THEORIES OF FINSLER SUBSPACES WITH THE STANDPOINT OF SEMI-SYMMETRIC CONNECTIONS

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The properties of semi-symmetric metric connection in Riemannian space have been studied by Yano (1970) and Imai (1972). In a recent paper Singh and Chaubey (1978) have deduced the Gauss-Codazzi equations in a subspace of Finsler space with the standpoint of semi-symmetric intrinsic connections. In this paper we shall study the geometrical properties of equations of Gauss deduced by Singh and Chaubey (1978). Further, we shall derive a condition for a umbilical subspace immersed in a Finsler space of constant curvature to be a space of constant curvature.

1. INTRODUCTION

Suppose F_n is a n-dimensional Finsler space having $g_{ij}(x, \dot{x})$ as its metric tensor and let $\Gamma^{*i}_{jk}(x, \dot{x})$ (or $P^{*i}_{jk}(x, \dot{x})$ be the connection parameters associated with Rund's process of δ-differentiation (Rund 1959, p. 81). The semi-symmetric connection parameter $\bar{\Gamma}^i_{jk}(x, \dot{x})$ is given by

$$\bar{\Gamma}^i_{jk} = \overset{*}{\Gamma}{}^i_{jk} + \delta^i_j P_k - g_{jk} P^i \qquad ...(1.1)$$

where $P^i(x, \dot{x})$ is an arbitrary vector field,

$P_k(x, \dot{x}) = g_{ik}(x, \dot{x}) P^i$ and δ^i_j are the Kronecker deltas :

Consider a subspace $F_m : x^i = x^i(u^\alpha)$, $(\alpha = 1, ..., m\ ;\ m < n)$ of the space F_n. A semi-symmetric intrinsic connection parameter $'\bar{\Gamma}^\alpha_{\beta\gamma}$ is defined by

$$'\bar{\Gamma}^\alpha_{\beta\gamma} = {}'\Gamma^\alpha_{\beta\gamma} + \delta^\alpha_\beta P_\gamma - g_{\beta\gamma} P^\alpha \qquad ...(1.2)$$

where $'\Gamma^\alpha_{\beta\gamma}$ is intrinsic connection co-efficient defined by Rund (1965), $g_{\alpha\beta}$ is metric tensor of the subspace and

$$P_\gamma = B^k_\gamma P_k \quad \text{and} \quad P^\alpha = g^{\alpha\gamma} P_\gamma. \qquad ...(1.3)$$

Throughout this paper the vector field P^i will be regarded tangent to the subspace so that the relation

$$P^i = B^i_\beta P^\beta, \quad (B^i_\beta = \partial x^i/\partial u^\beta)$$

will always hold.

The connection parameters $\bar{\Gamma}^i_{jk}$ and $'\bar{\Gamma}^\epsilon_{\alpha\beta}$ are used in defining the mixed derivative

$$T^i_{\alpha|\beta} = \frac{\partial T^i_\alpha}{\partial u^\beta} - \frac{\partial T^i_\alpha}{\partial \dot{u}^\lambda}\, '\bar{\Gamma}^\lambda_{\sigma\beta}\, \dot{u}^\sigma - T^i_\epsilon\, '\bar{\Gamma}^\epsilon_{\alpha\beta} + T^j_\alpha\, \bar{\Gamma}^i_{jk} B^k_\beta . \qquad ...(1.4)$$

The Cartan's induced and intrinsic connection coefficients $\overset{*}{\Gamma}{}^\epsilon_{\alpha\beta}$ and $'\Gamma^\epsilon_{\alpha\beta}$ (Rund 1959, p. 213) are related by

$$'\Gamma^\epsilon_{\alpha\beta} = \Lambda^\epsilon_{\alpha\beta} + \overset{*}{\Gamma}{}^\epsilon_{\alpha\beta} \qquad ...(1.5)$$

where

$$g_{\epsilon\gamma} \Lambda^\epsilon_{\alpha\beta} = \Lambda_{\alpha\gamma\beta} = \sum_\mu \big(\underset{(\mu)}{M}_{\beta\gamma}\, \underset{(\mu)}{\Omega}_{\alpha\sigma} + \underset{(\mu)}{M}_{\alpha\gamma}\, \underset{(\mu)}{\Omega}_{\beta\sigma} - \underset{(\mu)}{M}_{\alpha\beta}\, \underset{(\mu)}{\Omega}_{\gamma\sigma}\big)\, \dot{u}^\sigma$$

$$- \sum \big(\underset{(\mu)}{M}_{\lambda\alpha}\, C^\lambda_{\beta\gamma} + \underset{(\mu)}{M}_{\lambda\beta}\, C^\lambda_{\alpha\gamma} - \underset{(\mu)}{M}_{\lambda\gamma}\, C^\lambda_{\beta\alpha}\big)\, \underset{(\mu)}{\Omega}_{\sigma\theta}\, \dot{u}^\sigma \dot{u}^\theta,$$

$$\underset{(\mu)}{M}_{\alpha\beta} = \underset{(\mu)}{M}_{ij} B^i_\alpha B^j_\beta, \quad \underset{(\mu)}{M}_{jk} = C_{ijk} \underset{(\mu)}{N}{}^i \qquad ...(1.6)$$

and N_i and $\underset{(\mu)}{\Omega}_{\alpha\beta}$ are the components of the unit normal vector and the second fundamental tensor of the subspace.

Here

$$\underset{(\mu)}{k_n}(u, \dot{u}) = \frac{\underset{(\mu)}{\Omega}_{\sigma\lambda}(u, \dot{u})\, \dot{u}^\sigma \dot{u}^\lambda}{g_{\sigma\lambda}(u, \dot{u})\, \dot{u}^\sigma \dot{u}^\lambda} \qquad ...(1.7)$$

represents the normal curvature of F_m in the direction of $\dot{u}^\sigma$. Rund (1965) has defined following relations :

$$\text{(a)} \quad J^i_{\alpha\beta} = B^i_{\alpha\beta} - B^i_\epsilon\, '\Gamma^\epsilon_{\alpha\beta} + \overset{*}{\Gamma}{}^i_{jk} B^j_\alpha B^k_\beta$$

$$\text{(b)} \quad J^i_{\alpha\beta} = I^i_{\alpha\beta} - B^i_\epsilon \Lambda^\epsilon_{\alpha\beta} \qquad ...(1.8)$$

where

$$I^i_{\alpha\beta} = \sum_\mu \underset{(\mu)}{N}{}^i \underset{(\mu)}{\Omega}_{\alpha\beta} = B^i_{\alpha\beta} - B^i_\epsilon \overset{*}{\Gamma}{}^\epsilon_{\alpha\beta} + \overset{*}{\Gamma}{}^i_{jk} B^j_\alpha B^k_\beta .$$

The following equation of Gauss has been obtained by Singh and Chaubey (1978) :

$$
\begin{aligned}
\bar{K}_{\alpha\beta\gamma\varepsilon} = {} & \bar{K}_{ijhk}\, B^i_\alpha B^j_\beta B^h_\gamma B^k_\varepsilon + \sum_\nu \left(\underset{(\nu)}{\Omega}_{\alpha\gamma}\, \underset{(\nu)}{\Omega}_{\beta\varepsilon} - \underset{(\nu)}{\Omega}_{\alpha\varepsilon}\, \underset{(\nu)}{\Omega}_{\beta\gamma}\right) \\
& + 2\sum_\nu \sum_\mu \underset{(\mu,\nu)}{M}_\beta \left(\underset{(\nu)}{\Omega}_{\alpha\gamma}\, \underset{(\mu)}{\Omega}_{\varepsilon\sigma} - \underset{(\nu)}{\Omega}_{\alpha\varepsilon}\, \underset{(\mu)}{\Omega}_{\gamma\sigma}\right) \dot{u}^\sigma - 2\sum_\nu \underset{(\nu)}{M}_{\beta\lambda}\, P^\lambda \\
& \left(\underset{(\nu)}{\Omega}_{\alpha\varepsilon}\, g_{\gamma\sigma} - \underset{(\nu)}{\Omega}_{\alpha\gamma}\, g_{\varepsilon\sigma}\right) \dot{u}^\sigma - 2\sum_\nu \underset{(\nu)}{M}_{\lambda\beta} \left(\Lambda^\lambda_{\sigma\gamma}\, \underset{(\nu)}{\Omega}_{\alpha\varepsilon} - \Lambda^\lambda_{\sigma\varepsilon}\, \underset{(\nu)}{\Omega}_{\alpha\gamma}\right) \dot{u}^\sigma \\
& + \frac{\partial \bar{\Gamma}^i_{hk}}{\partial \dot{x}^l}\, B^h_\alpha\, g_{ij}\, B^j_\beta\, \dot{u}^\sigma \Big[\sum_\nu \left(\underset{(\nu)}{\Omega}_{\sigma\varepsilon} B^k_\gamma - \underset{(\nu)}{\Omega}_{\sigma\gamma} B^k_\varepsilon\right) \underset{(\nu)}{N^l} \\
& - B^l_\lambda \left(\Lambda^\lambda_{\sigma\varepsilon} B^k_\gamma - \Lambda^\lambda_{\sigma\gamma} B^k_\varepsilon\right)\Big] \\
& + \left(\Lambda_{\alpha\beta\varepsilon} P_\gamma - \Lambda_{\alpha\beta\gamma} P_\varepsilon\right) + \left(\Lambda_{\lambda\beta\gamma}\Lambda^\lambda_{\alpha\varepsilon} - \Lambda_{\lambda\beta\varepsilon}\Lambda^\lambda_{\alpha\gamma}\right) \\
& - g_{\delta\beta}\left(\Lambda^\delta_{\alpha\varepsilon|\gamma} - \Lambda^\delta_{\alpha\gamma|\varepsilon}\right) \qquad \ldots(1.9)
\end{aligned}
$$

where

$$\underset{(\mu,\nu)}{M}_\beta = \underset{(\mu)}{M}_{jl}\, B^j_\beta\, \underset{(\nu)}{N^l}$$

and $\bar{K}_{\alpha\beta\gamma\varepsilon}$ and $\bar{K}_{ijhk}$ are the curvature tensors of F_m and F_n respectively.

2. Riemannian Curvature of a Subspace

Consider a 2-direction given by the pair of vectors

$$\dot{x}^i = B^i_\alpha\, \dot{u}^\alpha \quad \text{and} \quad X^i = B^i_\alpha\, U^\alpha \qquad \ldots(2.1)$$

at a point P of the subspace F_m. For this 2-direction the Riemannian curvatures corresponding to semi-symmetric connection parameters $\bar{\Gamma}^i_{hk}$ and $'\bar{\Gamma}^\alpha_{\beta\gamma}$ of F_n and F_m are given by

$$\bar{K}(x, \dot{x}, X) = \frac{\bar{K}_{ijhk}(x, \dot{x})\, \dot{x}^i \dot{x}^h X^j X^k}{[g_{ih}(x, \dot{x})\, g_{jk}(x, \dot{x}) - g_{ij}(x, \dot{x})\, g_{hk}(x, \dot{x})]\, \dot{x}^i \dot{x}^h X^j X^k} \qquad \ldots(2.2)$$

$$\bar{\bar{K}}(u, \dot{u}, U) = \frac{\bar{K}_{\alpha\beta\gamma\varepsilon}(u, \dot{u})\, \dot{u}^\alpha \dot{u}^\gamma U^\beta U^\varepsilon}{[g_{\alpha\gamma}(u, \dot{u})\, g_{\beta\varepsilon}(u, \dot{u}) - g_{\alpha\beta}(u, \dot{u})\, g_{\gamma\varepsilon}(u, \dot{u})]\, \dot{u}^\alpha \dot{u}^\gamma U^\beta U^\varepsilon}. \qquad \ldots(2.3)$$

Multiplying (1.9) by $\dot{u}^\alpha \dot{u}^\gamma$, putting $F^2(u, \dot{u}) = g_{\alpha\gamma}(u, \dot{u})\, \dot{u}^\alpha \dot{u}^\gamma$ and using the relation (1.7) and the facts

$$\text{(a)}\quad \Lambda^{\varepsilon}_{\alpha\gamma}\, \dot u^{\alpha}\, \dot u^{\gamma} = 0, \quad \text{(b)}\quad \dot u^{\alpha}_{|\beta} = 0 \quad \text{and} \quad \text{(c)}\quad g_{\alpha\beta|\gamma} = 0 \qquad ...(2.4)$$

we get

$$\begin{aligned}
\bar K_{\alpha\beta\gamma\varepsilon}\, \dot u^{\alpha}\, \dot u^{\gamma} &= \bar K_{ijhk}\, \dot x^{i}\, \dot x^{h}\, B^{j}_{\beta}\, B^{k}_{\varepsilon} \\
&+ \sum_{\nu} (\underset{(\nu)}{k_n} F^2 \underset{(\nu)}{\Omega_{\beta\varepsilon}} - \underset{(\nu)}{\Omega_{\beta\gamma}} \underset{(\nu)}{\Omega_{\alpha\varepsilon}}\, \dot u^{\alpha}\, \dot u^{\gamma}) \\
&+ 2 \sum_{\nu} \underset{(\nu)}{M_{\lambda\beta}}\, \Lambda^{\lambda}_{\sigma\varepsilon} \underset{(\nu)}{k_n} F^2 \dot u^{\sigma} + 2 \sum_{\nu} \sum_{\mu} \underset{(\nu,\mu)}{M_{\beta}} \\
&\quad (\underset{(\nu)}{k_n} \underset{(\mu)}{\Omega_{\varepsilon\sigma}}\, \dot u^{\sigma} - \underset{(\mu)}{k_n} \underset{(\nu)}{\Omega_{\alpha\varepsilon}}\, \dot u^{\alpha}) F^2 - 2 \sum_{\nu} \underset{(\nu)}{M_{\beta\lambda}}\, P^{\lambda} \\
&\quad (\underset{(\nu)}{\Omega_{\alpha\varepsilon}}\, \dot u^{\alpha} - \underset{(\nu)}{k_n}\, g_{\varepsilon\sigma}\, \dot u^{\sigma}) F^2 + \frac{\partial \bar\Gamma^{i}_{hk}}{\partial \dot x^{l}}\, \dot u^{\sigma}\, \dot x^{h}\, g_{ij}\, B^{j}_{\beta} \\
&\quad [\sum_{\nu} (\underset{(\nu)}{\Omega_{\sigma\varepsilon}}\, \dot x^{k} - \underset{(\nu)}{\Omega_{\sigma\gamma}}\, B^{k}_{\varepsilon}\, \dot u^{\gamma}) \underset{(\nu)}{N^{l}} - B^{l}_{\lambda}\, \Lambda^{\lambda}_{\sigma\varepsilon}\, \dot x^{k}] \\
&+ \Lambda_{\alpha\beta\varepsilon} P_{\gamma}\, \dot u^{\alpha}\, \dot u^{\gamma} + \Lambda_{\lambda\beta\gamma}\, \Lambda^{\lambda}_{\alpha\varepsilon}\, \dot u^{\alpha}\, \dot u^{\gamma} - \Lambda_{\alpha\beta\varepsilon|\gamma}\, \dot u^{\alpha}\, \dot u^{\gamma}.
\end{aligned} \qquad ...(2.5)$$

Now using the relations (Singh 1978)

$$\frac{\partial \bar\Gamma^{i}_{hk}}{\partial \dot x^{l}}\, \dot x^{h}\, \dot x^{k} = 0 \qquad ...(2.6)$$

and

$$\begin{aligned}
\frac{\partial \bar\Gamma^{i}_{hk}}{\partial \dot x^{l}}\, \dot x^{h} (g_{ij} B^{j}_{\beta}\, B^{k}_{\varepsilon}) \underset{(\nu)}{k_n} F^2 \underset{(\nu)}{N^{l}} &= \overset{*}{C}{}^{i}_{kl|r}\, \dot x^{r}\, g_{ij}\, B^{j}_{\beta}\, B^{k}_{\varepsilon} \underset{(\nu)}{k_n} F^2 \underset{(\nu)}{N^{l}} \\
&+ 2 \underset{(\nu)}{M_{\varepsilon\theta}}\, g_{\beta\sigma}\, P^{\theta}\, \dot u^{\sigma} \underset{(\nu)}{k_n} F^2
\end{aligned} \qquad ...(2.7)$$

in (2.5), we get

$$\begin{aligned}
\bar K_{\alpha\beta\gamma\varepsilon}\, \dot u^{\alpha}\, \dot u^{\gamma} &= \bar K_{ijhk}\, \dot x^{i}\, \dot x^{h}\, B^{j}_{\beta}\, B^{k}_{\varepsilon} + \sum_{\nu} (\underset{(\nu)}{k_n} F^2 \underset{(\nu)}{\Omega_{\beta\varepsilon}} - \underset{(\nu)}{\Omega_{\beta\gamma}} \underset{(\nu)}{\Omega_{\alpha\varepsilon}}\, \dot u^{\alpha}\, \dot u^{\gamma}) \\
&+ 2 \sum_{\nu} \underset{(\nu)}{M_{\lambda\beta}}\, \Lambda^{\lambda}_{\sigma\varepsilon} \underset{(\nu)}{k_n} F^2 \dot u^{\sigma} + 2 \sum_{\nu} \sum_{\mu} \underset{(\nu,\mu)}{M_{\beta}} (\underset{(\nu)}{k_n} \underset{(\mu)}{\Omega_{\varepsilon\sigma}}\, \dot u^{\sigma} \\
&- \underset{(\mu)}{k_n} \underset{(\nu)}{\Omega_{\alpha\varepsilon}}\, \dot u^{\alpha}) F^2 - 2 \sum_{\nu} \underset{(\nu)}{M_{\beta\lambda}}\, P^{\lambda} (\underset{(\nu)}{\Omega_{\alpha\varepsilon}}\, \dot u^{\alpha} - \underset{(\nu)}{k_n}\, g_{\varepsilon\sigma}\, \dot u^{\sigma}) F^2 -
\end{aligned}$$

(equation continued on p. 116)

*The symbol '|' stands for Cartan's process of covariant differentiation.

$$- \sum_{\nu} C^{i}_{kl\,|\,r}\, \dot{x}^{r}\, g_{ij}\, B^{j}_{\beta}\, B^{k}_{\varepsilon}\, \underset{(\nu)}{k_n}\, F^2\, \underset{(\nu)}{N^l} - 2\sum_{\nu} \underset{(\nu)}{M_{\varepsilon\theta}}\, g_{\beta\sigma}\, P^{\theta}\, \dot{u}^{\sigma}\, \underset{(\nu)}{k_n}\, F^2$$

$$+ \Lambda_{\alpha\beta\varepsilon}\, P_{\gamma}\, \dot{u}^{\alpha}\, \dot{u}^{\gamma} + \Lambda_{\lambda\beta\gamma}\, \Lambda^{\lambda}_{\alpha\varepsilon}\, \dot{u}^{\alpha}\, \dot{u}^{\gamma} - \Lambda_{\alpha\beta\varepsilon|\gamma}\, \dot{u}^{\alpha}\, \dot{u}^{\gamma}. \qquad \ldots(2.8)$$

Multiplying (2.8) by $U^{\beta}\, U^{\varepsilon}$, dividing the resulting equation by

$$(g_{ih}\, g_{jk} - g_{ij}\, g_{hk})\, \dot{x}^{i}\, \dot{x}^{h}\, X^{j}\, X^{k} = 1 - (g_{\alpha\beta}\, \dot{u}^{\alpha}\, U^{\beta})^2$$

and using (2.2), (2.3) and normalising condition

$$F^2(x, \dot{x}) = g_{ij}(x, \dot{x})\, \dot{x}^{i}\, \dot{x}^{j} = 1$$

$$g_{ij}(x, \dot{x})\, X^{i}\, X^{j} = 1$$

we obtain

$$\bar{\bar{K}} = \bar{K} + \frac{\sum_{\nu} \underset{(\nu)}{k_n}\, \underset{(\nu)}{\bar{k}_n}}{1 - (g_{\alpha\beta}\, \dot{u}^{\alpha}\, U^{\beta})^2} - \frac{\sum_{\nu} (\underset{(\nu)}{\Omega_{\alpha\varepsilon}}\, \dot{u}^{\alpha}\, U^{\varepsilon})^2}{1 - (g_{\alpha\beta}\, \dot{u}^{\alpha}\, U^{\beta})^2}$$

$$+ \frac{2\sum_{\nu} \underset{(\nu)}{M_{\lambda\beta}}\, \Lambda^{\lambda}_{\sigma\varepsilon}\, \underset{(\nu)}{k_n}\, \dot{u}^{\sigma}\, U^{\beta}\, U^{\varepsilon}}{1 - (g_{\alpha\beta}\, \dot{u}^{\alpha}\, U^{\beta})^2}$$

$$+ \frac{2\sum_{\nu}\sum_{\mu} \underset{(\nu\mu)}{M_{\beta}}\, (\underset{(\nu)}{k_n}\, \underset{(\mu)}{\Omega_{\varepsilon\sigma}}\, \dot{u}^{\sigma} - \underset{(\mu)}{k_n}\, \underset{(\nu)}{\Omega_{\alpha\varepsilon}}\, \dot{u}^{\alpha})\, U^{\beta}\, U^{\varepsilon}}{1 - (g_{\alpha\beta}\, \dot{u}^{\alpha}\, U^{\beta})^2}$$

$$- \frac{2\sum_{\nu} \underset{(\nu)}{M_{\lambda\beta}}\, P^{\lambda}\, U^{\beta}\, U^{\varepsilon}\, (\underset{(\nu)}{\Omega_{\alpha\varepsilon}}\, \dot{u}^{\alpha} - \underset{(\nu)}{k_n}\, g_{\varepsilon\sigma}\, \dot{u}^{\sigma})}{1 - (g_{\alpha\beta}\, \dot{u}^{\alpha}\, U^{\beta})^2}$$

$$- \frac{\sum_{\nu} C^{i}_{kl\,|\,r}\, \dot{x}^{r}\, g_{ij}\, X^{j}\, X^{k}\, \underset{(\nu)}{k_n}\, \underset{(\nu)}{N^l}}{1 - (g_{\alpha\beta}\, \dot{u}^{\alpha}\, U^{\beta})^2}$$

$$- \frac{2\sum_{\nu} \underset{(\nu)}{M_{\varepsilon\theta}}\, g_{\beta\sigma}\, P^{\theta}\, \dot{u}^{\sigma}\, U^{\beta}\, U^{\varepsilon}\, \underset{(\nu)}{k_n}}{1 - (g_{\alpha\beta}\, \dot{u}^{\alpha}\, U^{\beta})^2}$$

$$+ \frac{1}{[1 - (g_{\alpha\beta}\, \dot{u}^{\alpha}\, U^{\beta})^2} \Big\{ \Lambda_{\alpha\beta\varepsilon}\, P_{\gamma}\, \dot{u}^{\alpha}\, \dot{u}^{\gamma} - \Lambda_{\lambda\beta\gamma}\, \Lambda^{\lambda}_{\alpha\varepsilon}\, \dot{u}^{\alpha}\, \dot{u}^{\gamma}$$

$$- \Lambda_{\alpha\beta\varepsilon|\gamma}\, \dot{u}^{\alpha}\, \dot{u}^{\gamma} \Big\}\, U^{\beta}\, U^{\varepsilon} \qquad \ldots(2.9)$$

where

$$\underset{(\nu)}{\bar{k}_n} = \underset{(\nu)}{\Omega_{\alpha\beta}}(u, \dot{u})\, U^{\alpha}\, U^{\beta}. \qquad \ldots(2.10)$$

This equation represents a relation between the two Riemannian curvatures $\bar{K}$ and $\bar{\bar{K}}$.

Now it is assumed that $\dot{u}^\alpha$ is a principal direction relative to all normals $\underset{(\nu)}{N^i}$ and U^α is conjugate with respect to $\dot{u}^\alpha$. This implies

$$\underset{(\nu)}{\Omega_{\alpha\beta}}(u, \dot{u})\, \dot{u}^\alpha = \underset{(\nu)}{k_n}\, g_{\alpha\beta}(u, \dot{u})\, \dot{u}^\alpha,$$

$$\underset{(\nu)}{\Omega_{\alpha\beta}}(u, \dot{u})\, \dot{u}^\alpha\, U^\beta = 0 \text{ and } g_{\alpha\beta}(u, \dot{u})\, \dot{u}^\alpha\, U^\beta = 0.$$

The eqn. (2.9) will now reduce to

$$\bar{\bar{K}} = \bar{K} + \sum_\nu \underset{(\nu)}{k_n}\, \underset{(\nu)}{\bar{k}_n} - C^{\,i}_{kl\,|\,r}\, \dot{x}^\gamma g_{ij} X^j X^k \sum_\nu \underset{(\nu)}{k_n}\, \underset{(\nu)}{N^l}$$
$$+ 2 \sum_\nu \underset{(\nu)}{M_{\lambda\beta}}\, \Lambda^{\lambda}_{\sigma\varepsilon}\, \underset{(\nu)}{k_n}\, \dot{u}^\sigma U^\beta U^\varepsilon$$
$$+ [\Lambda_{\alpha\beta\varepsilon} P_\gamma + \Lambda_{\lambda\beta\gamma} \Lambda^{\lambda}_{\alpha\varepsilon} - \Lambda_{\alpha\beta\varepsilon|\gamma}]\, \dot{u}^\alpha \dot{u}^\gamma U^\beta U^\varepsilon. \qquad \ldots(2.11)$$

Further, if F_n is affinely connected then $C^{\,i}_{kl\,|\,r} = 0$ and the above equation gives

$$\bar{\bar{K}} - \bar{K} = \sum_\nu \underset{(\nu)}{k_n}\, \underset{(\nu)}{\bar{k}_n} + 2\sum_\nu \underset{(\nu)}{M_{\lambda\beta}}\, \Lambda^{\lambda}_{\sigma\varepsilon}\, \underset{(\nu)}{k_n}\, \dot{u}^\sigma U^\beta U^\varepsilon$$
$$+ [\Lambda_{\alpha\beta\varepsilon} P_\gamma + \Lambda_{\lambda\beta\gamma} \Lambda^{\lambda}_{\alpha\varepsilon} - \Lambda_{\alpha\beta\varepsilon|\gamma}]\, \dot{u}^\alpha \dot{u}^\gamma U^\beta U^\varepsilon. \qquad \ldots(2.12)$$

This proves the following :

Theorem 2.1 — The difference between Riemannian curvatures $\bar{\bar{K}}(u, \dot{u}, U)$ and $\bar{K}(x, \dot{x}, X)$ at a point P of subspace F_m of an affinely connected Finsler space F_n, with respect to a 2-direction determined by a principal direction $\dot{u}^\alpha$ and its conjugate U^α is given by the relation (2.12).

3. Ricci Tensors

Multiplying the eqn. (1.9) by $g^{\lambda\beta}$, using relations

$$\bar{K}^{\lambda}_{\alpha\gamma\varepsilon} = g^{\lambda\beta}\, \bar{K}_{\alpha\beta\gamma\varepsilon},\ \bar{K}^{l}_{ihk} = g^{lj}\, \bar{K}_{ijhk} \text{ and } g^{\lambda\beta} B^{j}_{\beta} = g^{lj} B^{\lambda}_{l}$$

we obtain

$$\bar{K}^{\lambda}_{\alpha\gamma\varepsilon} = \bar{K}^{l}_{ihk}\, B^{\lambda}_{l}\, B^{h}_{\gamma}\, B^{i}_{\alpha}\, B^{k}_{\varepsilon} + g^{\lambda\beta} \sum_\nu (\underset{(\nu)}{\Omega_{\alpha\gamma}}\, \underset{(\nu)}{\Omega_{\beta\varepsilon}} - \underset{(\nu)}{\Omega_{\alpha\varepsilon}}\, \underset{(\nu)}{\Omega_{\beta\gamma}})$$
$$+ 2 \sum_\nu \sum_\mu \underset{(\nu,\mu)}{M^{\lambda}}\, (\underset{(\nu)}{\Omega_{\alpha\gamma}}\, \underset{(\mu)}{\Omega_{\sigma\varepsilon}} - \underset{(\nu)}{\Omega_{\alpha\varepsilon}}\, \underset{(\mu)}{\Omega_{\gamma\sigma}})\, \dot{u}^\sigma -$$

(*equation continued on p.* 118)

$$- 2 \sum_{\nu} \underset{(\nu)}{M}{}^{\lambda}_{\theta} (\Lambda^{\theta}_{\sigma\gamma} \underset{(\nu)}{\Omega}_{\alpha\varepsilon} - \Lambda^{\theta}_{\sigma\varepsilon} \underset{(\nu)}{\Omega}_{\alpha\gamma}) \dot{u}^{\sigma} - 2\sum_{\nu} \underset{(\nu)}{M}{}^{\lambda}_{\theta} P^{\theta}$$

$$(\underset{(\nu)}{\Omega}_{\alpha\varepsilon} g_{\gamma\sigma} - \underset{(\nu)}{\Omega}_{\alpha\gamma} g_{\varepsilon\sigma}) \dot{u}^{\sigma} + \frac{\partial \overline{\Gamma}^{i}_{hk}}{\partial \dot{x}^{l}} B^{h}_{\alpha} B^{\lambda}_{i}$$

$$[\sum_{\nu} (\underset{(\nu)}{\Omega}_{\sigma\varepsilon} B^{k}_{\gamma} - \underset{(\nu)}{\Omega}_{\sigma\gamma} B^{k}_{\varepsilon}) \underset{(\nu)}{N}{}^{l} - B^{l}_{\theta} (\Lambda^{\theta}_{\sigma\varepsilon} B^{k}_{\gamma} - \Lambda^{\theta}_{\sigma\gamma} B^{k}_{\varepsilon})] \dot{u}^{\sigma}$$

$$+ (\Lambda^{\lambda}_{\alpha\varepsilon} P_{\gamma} - \Lambda^{\lambda}_{\alpha\gamma} P_{\varepsilon}) + (\Lambda^{\lambda}_{\theta\gamma} \Lambda^{\theta}_{\alpha\varepsilon} - \Lambda^{\lambda}_{\theta\varepsilon} \Lambda^{\theta}_{\alpha\gamma})$$

$$- (\Lambda^{\lambda}_{\alpha\gamma|\varepsilon} - \Lambda^{\lambda}_{\alpha\varepsilon|\gamma}). \qquad \ldots(3.1)$$

Contracting λ and ϵ in above and using identities

$$B^{\varepsilon}_{l} B^{k}_{\varepsilon} = \delta^{k}_{l} - \sum_{\nu} \underset{(\nu)}{N}{}^{k} \underset{(\nu)}{N}_{l}, \quad \underset{(\nu)}{M}{}^{\varepsilon}_{\theta} g_{\varepsilon\sigma} \dot{u}^{\sigma} = 0$$

we get

$$\overline{K}_{\alpha\gamma} = (\overline{K}_{ih} - \sum_{\nu} \overline{K}^{l}_{ihk} \underset{(\nu)}{N}{}^{k} \underset{(\nu)}{N}_{l}) B^{i}_{\alpha} B^{h}_{\gamma} + \sum_{\nu} (\underset{(\nu)}{M} \underset{(\nu)}{\Omega}_{\alpha\gamma} - \underset{(\nu)}{\Omega}_{\alpha\varepsilon} g^{\beta\varepsilon} \underset{(\nu)}{\Omega}_{\beta\gamma})$$

$$+ 2\sum_{\nu} \sum_{\mu} \underset{(\nu,\mu)}{M}{}^{\varepsilon} (\underset{(\nu)}{\Omega}_{\alpha\gamma} \underset{(\mu)}{\Omega}_{\sigma\varepsilon} - \underset{(\nu)}{\Omega}_{\alpha\varepsilon} \underset{(\mu)}{\Omega}_{\gamma\sigma}) \dot{u}^{\sigma} - 2\sum_{\nu} \underset{(\nu)}{M}{}^{\varepsilon}_{\theta}$$

$$(\Lambda^{\theta}_{\sigma\gamma} \underset{(\nu)}{\Omega}_{\alpha\varepsilon} - \Lambda^{\theta}_{\sigma\varepsilon} \underset{(\nu)}{\Omega}_{\alpha\gamma}) \dot{u}^{\sigma} - 2\sum_{\nu} \underset{(\nu)}{M}{}^{\varepsilon}_{\theta} P^{\theta} \underset{(\nu)}{\Omega}_{\alpha\varepsilon} g_{\gamma\sigma} \dot{u}^{\sigma}$$

$$+ \frac{\partial \overline{\Gamma}^{i}_{hk}}{\partial \dot{x}^{l}} B^{h}_{\alpha} B^{\varepsilon}_{i} [\sum_{\nu} (\underset{(\nu)}{\Omega}_{\sigma\varepsilon} B^{k}_{\gamma} - \underset{(\nu)}{\Omega}_{\sigma\gamma} B^{k}_{\varepsilon}) \underset{(\nu)}{N}{}^{l} - B^{l}_{\theta} (\Lambda^{\theta}_{\sigma\varepsilon} B^{k}_{\gamma}$$

$$- \Lambda^{\theta}_{\sigma\gamma} B^{k}_{\varepsilon})] \dot{u}^{\sigma} + (\Lambda^{\varepsilon}_{\alpha\varepsilon} P_{\gamma} - \Lambda^{\varepsilon}_{\alpha\gamma} P_{\varepsilon}) + (\Lambda^{\varepsilon}_{\theta\gamma} \Lambda^{\theta}_{\alpha\varepsilon} - \Lambda^{\varepsilon}_{\alpha\varepsilon} \Lambda^{\theta}_{\alpha\gamma})$$

$$+ (\Lambda^{\varepsilon}_{\alpha\gamma|\varepsilon} - \Lambda^{\varepsilon}_{\alpha\varepsilon|\gamma}) \qquad \ldots(3.2)$$

where

$$\underset{(\nu)}{M} = g^{\beta\varepsilon} \underset{(\nu)}{\Omega}_{\beta\varepsilon} \quad \text{and} \quad \overline{K}_{\alpha\gamma} = \overline{K}^{\varepsilon}_{\alpha\gamma\varepsilon}, \overline{K}_{ih} = \overline{K}^{k}_{ihk}$$

are Ricci tensors of F_m and F_n corresponding to curvature tensors $\overline{K}^{\lambda}_{\alpha\beta\varepsilon}$ and $\overline{K}^{l}_{ihk}$ respectively.

If $\dot{u}^{\alpha}$ is a principal direction relative to all the normals of the subspace, then

$$\underset{(\nu)}{\Omega}_{\alpha\varepsilon}(u, \dot{u}) \dot{u}^{\alpha} = \underset{(\nu)}{k_n} g_{\alpha\varepsilon}(u, \dot{u}) \dot{u}^{\alpha} \qquad \ldots(3.3)$$

and

$$\text{(a)}\ \underset{(\nu,\mu)}{M^{\varepsilon}}\ \underset{(\mu)}{\Omega_{\alpha\varepsilon}}\ \dot{u}^{\alpha} = 0, \quad \text{(b)}\ B^{\varepsilon}_{i}\ \underset{(\nu)}{\Omega_{\varepsilon\sigma}}\ \dot{u}^{\sigma} = \underset{(\nu)}{k_n}\ g_{ij}\ \dot{x}^{j}. \qquad \ldots(3.4)$$

Also we have

$$g^{\beta\varepsilon}\ \underset{(\nu)}{\Omega_{\alpha\varepsilon}}\ \underset{(\nu)}{\Omega_{\beta\gamma}}\ \dot{u}^{\alpha} = \underset{(\nu)}{k^{2}_{n}}\ g_{\alpha\gamma}\ \dot{u}^{\alpha},\ \underset{(\nu)}{M^{\varepsilon}_{\theta}}\ \underset{(\nu)}{\Omega_{\alpha\varepsilon}}\ \dot{u}^{\alpha} = 0 \qquad \ldots(3.5)$$

Multiplying eqn. (3.2) by $\dot{u}^{\alpha}$ and using (3.3), (3.4*a*), (3.5) and the relations (Singh 1978)

$$B^{\varepsilon}_{i}\ \frac{\partial \bar{\Gamma}^{i}_{hk}}{\partial \dot{x}^{l}}\ \dot{x}^{h} \sum_{\nu} (\underset{(\nu)}{\Omega_{\varepsilon\sigma}}\ B^{k}_{\gamma} - \underset{(\nu)}{\Omega_{\sigma\gamma}}\ B^{k}_{\varepsilon})\ \underset{(\nu)}{N^{l}}\ \dot{u}^{\sigma}$$

$$= -\ C^{i}_{kl\,|\,r}\ \dot{x}^{r}\ g_{\sigma\gamma}\ \dot{u}^{\sigma}\ B^{k}_{\varepsilon}\ \ B^{\varepsilon}_{i} \sum_{\nu} \underset{(\nu)}{k_n}\ \underset{(\nu)}{N^{l}} + 2F^{2} \sum_{\nu} \underset{(\nu)}{M_{\theta\gamma}}\ P^{\theta}\ \underset{(\nu)}{k_n} \qquad \ldots(3.6)$$

we obtain

$$\bar{K}_{\alpha\gamma}\ \dot{u}^{\alpha} = (\bar{K}_{ih}\ \dot{x}^{h} - \sum_{\nu} \bar{K}^{l}_{ihk}\ \dot{x}^{i}\ \underset{(\nu)}{N^{k}}\ \underset{(\nu)}{N_{l}})\ B^{h}_{\gamma} + \sum_{\nu} g_{\alpha\gamma}(u, \dot{u})\ \dot{u}^{\alpha}\ \underset{(\nu)}{k_n}$$

$$(\underset{(\nu)}{M} - \underset{(\nu)}{k_n} - C^{i}_{kl\,|\,r}\ \dot{x}^{r}\ \underset{(\nu)}{N^{l}}\ B^{k}_{\varepsilon}\ \ B^{\varepsilon}_{i}) + 2F^{2} \sum_{\nu} \underset{(\nu)}{M_{\theta\gamma}}\ P^{\theta}\ \underset{(\nu)}{k_n}$$

$$+\ 2\sum_{\nu} \underset{(\nu)}{M^{\varepsilon}_{\theta}}\ \Lambda^{\theta}_{\sigma\varepsilon}\ \underset{(\nu)}{k_n}\ g_{\alpha\gamma}\ \dot{u}^{\alpha}\ \dot{u}^{\sigma} - \frac{\partial \bar{\Gamma}^{i}_{hk}}{\partial \dot{x}^{l}}\ B^{\varepsilon}_{i}\ \ B^{h}_{\alpha}\ \ \dot{u}^{\alpha}\ B^{l}_{\theta}$$

$$(\Lambda^{\theta}_{\sigma\varepsilon}\ B^{k}_{\gamma} - \Lambda^{\theta}_{\sigma\gamma}\ B^{k}_{\varepsilon})\ \dot{u}^{\sigma} + (\Lambda^{\varepsilon}_{\alpha\varepsilon}\ P_{\gamma} - \Lambda^{\varepsilon}_{\alpha\gamma}\ P_{\varepsilon})\ \dot{u}^{\alpha}$$

$$+\ (\Lambda^{\varepsilon}_{\theta\gamma}\ \ \Lambda^{\theta}_{\alpha\varepsilon} - \Lambda^{\varepsilon}_{\theta\varepsilon}\ \Lambda^{\theta}_{\alpha\gamma})\ \dot{u}^{\alpha} + (\Lambda^{\varepsilon}_{\alpha\gamma|\varepsilon} - \Lambda^{\varepsilon}_{\alpha\varepsilon|\gamma})\ \dot{u}^{\alpha}. \qquad \ldots(3.7)$$

In analogy with the definition given in Rund (1959, p. 133), we shall call F_n to be a space of constant curvature with respect to $\bar{K}^{l}_{ihk}$ provided that

$$\bar{K}^{l}_{ihk}\ \dot{x}^{i} = \bar{K}(g_{ih}\ \delta^{l}_{k} - g_{ik}\ \delta^{l}_{h})\ \dot{x}^{i} \qquad \ldots(3.8)$$

where $\bar{K}$ is a constant. This equation yields

$$\bar{K}_{ih}\ \dot{x}^{i} = \bar{K}(n-1)\ g_{ih}\ B^{i}_{\alpha}\ \ \dot{u}^{\alpha} \qquad \ldots(3.9)$$

and

$$\sum_{\nu} \bar{K}^{l}_{ihk}\ \dot{x}^{i}\ \underset{(\nu)}{N^{k}}\ \underset{(\nu)}{N_{l}} = \bar{K}(n-m)\ g_{ih}\ \dot{x}^{i} = \bar{K}(n-m)\ g_{ih}\ B^{i}_{\alpha}\ \ \dot{u}^{\alpha}. \qquad \ldots(3.10)$$

After substituting these values in (3.7), we get

$$\bar{K}_{\alpha\gamma}\,\dot{u}^{\alpha} = \omega\, g_{\alpha\gamma}\,\dot{u}^{\alpha} + 2F^2 \sum_{\nu} \underset{(\nu)}{M}_{\theta\gamma}\, P^{\theta}\, \underset{(\nu)}{k_n} - \frac{\partial \bar{\Gamma}^{i}_{hk}}{\partial \dot{x}^{l}}\, B^{\epsilon}_{i}\, B^{h}_{\alpha}\, \dot{u}^{\alpha}\, B^{l}_{\theta}$$

$$(\Lambda^{\theta}_{\sigma\epsilon}\, B^{k}_{\gamma} - \Lambda^{\theta}_{\sigma\gamma}\, B^{k}_{\epsilon})\,\dot{u}^{\sigma} + (\Lambda^{\epsilon}_{\alpha\epsilon}\, P_{\gamma} - \Lambda^{\epsilon}_{\alpha\gamma}\, P_{\epsilon})\,\dot{u}^{\alpha}$$

$$+ (\Lambda^{\epsilon}_{\theta\gamma}\, \Lambda^{\theta}_{\alpha\epsilon} - \Lambda^{\epsilon}_{\alpha\epsilon}\, \Lambda^{\theta}_{\alpha\gamma})\,\dot{u}^{\alpha} + (\Lambda^{\epsilon}_{\alpha\gamma|\epsilon} - \Lambda^{\epsilon}_{\alpha\gamma|\epsilon})\,\dot{u}^{\alpha} \qquad \ldots(3.11)$$

where

$$\omega = \bar{K}(m-1) + \sum_{\nu} \underset{(\nu)}{k_n}\,(\underset{(\nu)}{M} - \underset{(\nu)}{k_n} - \underset{(\nu)}{M}{}^{\epsilon}_{\theta}\, \Lambda^{\theta}_{\sigma\epsilon}\, \dot{u}^{\sigma} - C^{i}_{kl\,|\,h}\, \dot{x}^{h}\, \underset{(\nu)}{N^{l}}\, B^{\epsilon}_{i}\, B^{k}_{\epsilon}). \qquad \ldots(3.12)$$

The direction $\dot{u}^{\alpha}$ will be called Ricci principal direction of the subspace with respect to the semi-symmetric intrinsic connection parameter if

$$\bar{K}_{\alpha\gamma}(u, \dot{u})\,\dot{u}^{\alpha} = \omega\, g_{\alpha\gamma}(u, \dot{u})\,\dot{u}^{\alpha}$$

ω being a scalar.

Equation (3.11) proves the following :

Theorem 3.1 — The necessary and sufficient condition that a principal direction $\dot{u}^{\alpha}$ of the subspace F_m $(m > 1)$ imbedded in a Finsler space F_n of constant curvature $\bar{K}$ be a Ricci principal direction with the associated scalar

$$\omega = \bar{K}(m-1) + \sum_{\nu} \underset{(\nu)}{k_n}\,(\underset{(\nu)}{M} - \underset{(\nu)}{k_n} - \underset{(\nu)}{M}{}^{\epsilon}_{\theta}\, \Lambda^{\theta}_{\sigma\epsilon}\, \dot{u}^{\sigma} - C^{i}_{kl\,|\,h}\, \dot{x}^{h}\, \underset{(\nu)}{N^{l}}\, B^{\epsilon}_{i}\, B^{k}_{\epsilon})$$

is that

$$2F^2 \sum_{\nu} \underset{(\nu)}{M}_{\theta\gamma}\, P^{\theta}\, \underset{(\nu)}{k_n} - \frac{\partial \bar{\Gamma}^{i}_{hk}}{\partial \dot{x}^{l}}\, B^{\epsilon}_{i}\, B^{h}_{\alpha}\, \dot{u}^{\alpha}\, B^{l}_{\theta}\, (\Lambda^{\theta}_{\sigma\epsilon}\, B^{k}_{\gamma} - \Lambda^{\theta}_{\sigma\gamma}\, B^{k}_{\epsilon})\,\dot{u}^{\sigma}$$

$$+ (\Lambda^{\epsilon}_{\alpha\epsilon}\, P_{\gamma} - \Lambda^{\epsilon}_{\alpha\gamma}\, P_{\epsilon})\,\dot{u}^{\alpha} + (\Lambda^{\epsilon}_{\theta\sigma}\, \Lambda^{\theta}_{\alpha\epsilon} - \Lambda^{\epsilon}_{\theta\epsilon}\, \Lambda^{\theta}_{\alpha\gamma})\,\dot{u}^{\alpha}$$

$$+ (\Lambda^{\epsilon}_{\alpha\gamma|\epsilon} - \Lambda^{\epsilon}_{\alpha\epsilon|\gamma})\,\dot{u}^{\alpha} = 0. \qquad \ldots(3.13)$$

If the space F_n is affinely connected then $C^{i}_{kl\,|\,r} = 0$ and the above theorem reduces to :

Theorem 3.2 — The necessary and sufficient condition that a principal direction $\dot{u}^{\alpha}$ of the subspace F_m $(m > 1)$ imbedded in an affinely connected Finsler space F_n of constant curvature $\bar{K}$ be a Ricci principal direction with the associated scalar

$$\alpha_1 = \bar{K}(m-1) + \sum_{\nu} \underset{(\nu)}{k_n} (\underset{(\nu)}{M} - \underset{(\nu)}{k_n} - \underset{(\nu)}{M^s_{\theta}} \Lambda^{\theta}_{\sigma s} \dot{u}^{\sigma})$$

is that eqn. (3.13) holds.

4. Umbilical Subspaces

An unbilical subspace is characterised by

$$\underset{(\mu)}{\Omega_{\alpha\beta}} = \underset{(\mu)}{k_n}\, g_{\alpha\beta}. \qquad \ldots(4.1)$$

This equation gives

$$\frac{\partial \underset{(\mu)}{\Omega_{\alpha\beta}}}{\partial \dot{u}^{\lambda}} \dot{u}^{\alpha} = \frac{\partial \underset{(\nu)}{k_n}}{\partial \dot{u}_{\lambda}} g_{\alpha\beta}\, \dot{u}^{\alpha} \qquad \ldots(4.2)$$

and

$$\frac{\partial \underset{(\mu)}{\Omega_{\alpha\beta}}}{\partial \dot{u}^{\lambda}} \dot{u}^{\alpha}\, \dot{u}^{\beta} = \frac{\partial \underset{(\mu)}{k_n}}{\partial \dot{u}^{\lambda}} F^2. \qquad \ldots(4.3)$$

Now relations (1.8*a*) and (1.8*b*) give

$$\underset{(\mu)}{\Omega_{\alpha\beta}} = \underset{(\mu)}{N_i}\, J^i_{\alpha\beta} = \underset{(\mu)}{N_i}\, (B^i_{\alpha\beta} - B^i_s\, {}'\Gamma^s_{\alpha\beta} + \overset{*}{\Gamma}{}^i_{hk}\, B^h_{\alpha}\, B^k_{\beta}). \qquad \ldots(4.4)$$

Differentiating (4.4) with respect to $\dot{u}^{\lambda}$ and using relations (1.8*b*) and (Rund 1959, p. 81)

$$\frac{\partial \overset{*}{\Gamma}{}^i_{hk}}{\partial \dot{x}^l} \dot{x}^h = C^i_{kl\,|\,r}\, \dot{x}^r$$

we get

$$\frac{\partial \underset{(\mu)}{\Omega_{\alpha\beta}}}{\partial \dot{u}^{\lambda}} \dot{u}^{\alpha} = \frac{\partial \underset{(\mu)}{N_i}}{\partial \dot{u}^{\lambda}} \sum_{\nu} \underset{(\nu)}{\Omega_{\alpha\beta}}\, \underset{(\nu)}{N^i}\, \dot{u}^{\alpha} - \frac{\partial \underset{(\mu)}{N_i}}{\partial \dot{u}^{\lambda}} B^i_s\, \Lambda^s_{\alpha\beta}\, \dot{u}^{\alpha}$$

$$+ \underset{(\mu)}{N_i}\, C^i_{kl\,|\,h}\, \dot{x}^h B^l_{\lambda}\, B^k_{\beta}$$

which in view of the fact $\frac{\partial \underset{(\mu)}{N_i}}{\partial \dot{u}^{\lambda}} B^i_s = 0$ gives

$$\frac{\partial \underset{(\mu)}{\Omega_{\alpha\beta}}}{\partial \dot{u}^{\lambda}} \dot{u}^{\alpha} = \frac{\partial \underset{(\mu)}{N_i}}{\partial \dot{u}^{\lambda}} \sum_{\nu} \underset{(\nu)}{\Omega_{\alpha\beta}}\, \underset{(\nu)}{N^i}\, \dot{u}^{\alpha} + \underset{(\mu)}{N_i}\, C^i_{kl\,|\,h}\, \dot{x}^h\, B^l_{\lambda}\, B^k_{\beta}\,. \qquad \ldots(4.5)$$

Multiplying (4.5) by $\dot{u}^{\beta}$ and simplifying the resulting equation with the help of identities

$$C^{i}_{kl\,|\,h}\,\dot{x}^{h}\,\dot{x}^{k}=0 \quad\text{and}\quad \underset{(\nu)}{\Omega_{\alpha\beta}}\,\dot{u}^{\alpha}\,\dot{u}^{\beta}=\underset{(\nu)}{k_n}\,F^{2}$$

we get

$$\frac{\partial \underset{(\mu)}{\Omega_{\alpha\beta}}}{\partial \dot{u}^{\lambda}}\,\dot{u}^{\alpha}\,\dot{u}^{\beta}=\frac{\partial \underset{(\mu)}{N_{i}}}{\partial \dot{u}^{\lambda}}\sum_{\nu}F^{2}\,\underset{(\nu)}{k_n}\,\underset{(\nu)}{N^{i}}. \qquad \text{...(4.6)}$$

A comparison of (4.3) and (4.6) will give

$$\sum_{\nu}\frac{\partial \underset{(\mu)}{N_{i}}}{\partial \dot{u}^{\lambda}}\,\underset{(\nu)}{k_n}\,\underset{(\nu)}{N^{i}}=\frac{\partial \underset{(\mu)}{k_n}}{\partial \dot{u}^{\lambda}}. \qquad \text{...(4.7)}$$

The substitution from (4.1) and (4.7) in (4.5) gives

$$\frac{\partial \underset{(\mu)}{\Omega_{\alpha\beta}}}{\partial \dot{u}^{\lambda}}\,\dot{u}^{\alpha}=\frac{\partial \underset{(\mu)}{k_n}}{\partial \dot{u}^{\lambda}}\,g_{\alpha\beta}\,\dot{u}^{\alpha}+\underset{(\mu)}{N_{i}}\,C^{i}_{kl\,|\,h}\,\dot{x}^{h}\,B^{l}_{\lambda}\,B^{k}_{\beta}$$

which after comparison with (4.2) gives

$$\underset{(\mu)}{N_{i}}\,C^{i}_{kl\,|\,h}\,\dot{x}^{h}\,B^{l}_{\lambda}\,B^{k}_{\beta}=0. \qquad \text{...(4.8)}$$

On the other hand a direct calculation gives

$$g_{ij}\,B^{j}_{\beta}\,B^{k}_{\varepsilon}\,C^{i}_{kl\,|\,h}\,\dot{x}^{h}\,\underset{(\mu)}{N^{l}}=C_{kjl\,|\,h}\,\dot{x}^{h}\,\underset{(\mu)}{N^{l}}\,B^{j}_{\beta}\,B^{k}_{\varepsilon}$$

$$=\underset{(\mu)}{N_{i}}\,C^{i}_{kl\,|\,h}\,\dot{x}^{h}\,B^{l}_{\beta}\,B^{k}_{\varepsilon}\,. \qquad \text{...(4.9)}$$

Substituting (4.1), (4.8) and (4.9) in (2.7), we get

$$\bar{K}_{\alpha\beta\gamma\varepsilon}\,\dot{u}^{\alpha}\,\dot{u}^{\gamma}=\bar{K}_{ijhk}\,\dot{x}^{i}\,\dot{x}^{h}\,B^{j}_{\beta}\,B^{k}_{\varepsilon}+\sum_{\nu}\underset{(\nu)}{k_n^{2}}\,(F^{2}\,g_{\beta\varepsilon}-g_{\alpha\varepsilon}\,g_{\beta\gamma}\,\dot{u}^{\alpha}\,\dot{u}^{\gamma})$$

$$+\,2\sum_{\nu}\underset{(\nu)}{M_{\lambda\beta}}\,\Lambda^{\lambda}_{\sigma\varepsilon}\,\underset{(\nu)}{k_n}\,F^{2}\,\dot{u}^{\sigma}-2\sum_{\nu}\underset{(\nu)}{M_{\varepsilon\theta}}\,g_{\beta\sigma}\,P^{\theta}\,\dot{u}^{\sigma}\,\underset{(\nu)}{k_n}\,F^{2}$$

$$+\,\Lambda_{\alpha\beta\varepsilon}\,P_{\gamma}\,\dot{u}^{\alpha}\,\dot{u}^{\gamma}+\Lambda_{\lambda\beta\gamma}\,\Lambda^{\lambda}_{\alpha\varepsilon}\,\dot{u}^{\alpha}\,\dot{u}^{\gamma}-\Lambda_{\alpha\beta\varepsilon|\gamma}\,\dot{u}^{\alpha}\,\dot{u}^{\gamma}. \qquad \text{...(4.10)}$$

Suppose in particular that F_n is a space of scalar curvature with respect to $\bar{K}^{l}_{ihk}$. This space is characterized by (Rund 1959, p. 133)

$$\bar{K}_{ijhk}\,\dot{x}^{i}\,\dot{x}^{h}=\bar{K}[F^{2}\,g_{jk}-(g_{ij}\,\dot{x}^{i})\,(g_{hk}\,\dot{x}^{h})]. \qquad \text{...(4.11)}$$

A simplification based on (4.11) and (4.10) gives

$$\bar{K}_{\alpha\beta\gamma\varepsilon}\, \dot{u}^{\alpha}\, \dot{u}^{\gamma} = (\bar{K} + \sum_{\nu} \underset{(\nu)}{k_n^2})\, [F^2 g_{\beta\varepsilon} - (g_{\alpha\beta}\, \dot{u}^{\alpha})\, (g_{\gamma\varepsilon}\, \dot{u}^{\gamma})]$$

$$+ 2\sum_{\nu} \underset{(\nu)}{M_{\lambda\beta}}\, \Lambda^{\lambda}_{\sigma\varepsilon}\, \underset{(\nu)}{k_n}\, F^2\, \dot{u}^{\sigma} - 2\sum_{\nu} \underset{(\nu)}{M_{\varepsilon\theta}}\, g_{\beta\sigma}\, P^{\theta}\, \dot{u}^{\sigma}\, \underset{(\nu)}{k_n}\, F^2$$

$$+ \Lambda_{\alpha\beta\varepsilon}\, P_{\gamma}\, \dot{u}^{\alpha}\, \dot{u}^{\gamma} + \Lambda_{\lambda\beta\gamma}\, \Lambda^{\lambda}_{\alpha\varepsilon}\, \dot{u}^{\alpha}\, \dot{u}^{\gamma} - \Lambda_{\alpha\beta\varepsilon|\gamma}\, \dot{u}^{\alpha}\, \dot{u}^{\gamma} \qquad \ldots(4.12)$$

which proves the following :

Theorem 4.1 — If a Finsler space of scalar curvature $\bar{K}$ admits an umbilical subspace F_n then the necessary and sufficient condition that the latter is a space of scalar curvature $(\bar{K} + \sum_{\nu} \underset{(\nu)}{k_n^2})$ is that

$$2\sum_{\nu} \underset{(\nu)}{M_{\lambda\beta}}\, \Lambda^{\lambda}_{\sigma\varepsilon}\, \underset{(\nu)}{k_n}\, F^2\, \dot{u}^{\sigma} - 2\sum_{\nu} \underset{(\nu)}{M_{\varepsilon\theta}}\, g_{\beta\sigma}\, P^{\theta}\, \dot{u}^{\sigma}\, \underset{(\nu)}{k_n}\, F^2 + \Lambda_{\alpha\beta\varepsilon}\, P_{\gamma}\, \dot{u}^{\alpha}\, \dot{u}^{\gamma}$$

$$+ \Lambda_{\lambda\beta\gamma}\, \Lambda^{\lambda}_{\alpha\varepsilon}\, \dot{u}^{\alpha}\, \dot{u}^{\gamma} - \Lambda_{\alpha\beta\varepsilon|\gamma}\, \dot{u}^{\alpha}\, \dot{u}^{\gamma} = 0. \qquad \ldots(4.13)$$

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