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BOUNDS ON THE CURVATURE OF LEVEL LINES UNDER CERTAIN CLASSES OF UNIVALENT AND LOCALLY UNIVALENT MAPPINGS

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For $\alpha < 1$, let S^*_α, C_α be the classes of regular analytic functions in the unit disc E which are, respectively, starlike of order α and convex of order α, and let K be the subclass of close-to-convex functions f defined by

$$\text{Re}\,(zf'(z) \mid \phi(z)) > 0, \phi \in S^*_0$$

and let $V_k, k \geqslant 2$ be the class of functions of bounded boundary rotation at most $k\pi$. In this paper sharp bounds on the curvature of the image of $|z| = r, 0 < r < 1$, under a mapping f belonging to the classes C_α, K and V_k have been obtained. The main tool is the interesting inequality, for $f \in S^*_0$, $|z| = r$, given by

$$(1 - r^2)\left|\frac{f(z)}{z}\right| \leqslant \text{Re}\,\frac{zf'(z)}{f(z)} \leqslant \frac{1 + r}{1 - r} + \frac{2r \log\left[(1 - r)^2\left|\frac{f(z)}{z}\right|\right]}{(1 - r^2)\log\frac{1 + r}{1 - r}}$$

whose elementary proof has been given.

1. INTRODUCTION AND STATEMENT OF RESULTS

Let A denote the class of functions regular analytic in the unit disc $E = \{z| \mid z \mid < 1\}$. We shall need the following subclasses of A:

(i) The subclass P of functions whose real part is positive in E and which have the value 1 at $z = 0$.

(ii) For $\alpha < 1$ the subclasses S^*_α, C_α, K and for $k \geqslant 2$ the class V_k of functions f which are normalized so that $f(0) = f'(0) - 1 = 0$ and which are defined by the following conditions :

$$f \in S^*_\alpha \text{ iff } z\frac{f'(z)}{f(z)} = (1 - \alpha)\,p(z) + \alpha,\ z \in E,\ p \in P \qquad ...(1)$$

$$f \in C_\alpha \text{ iff } 1 + z\frac{f''(z)}{f(z)} = (1 - \alpha)\,p(z) + \alpha,\ z \in E,\ p \in P \qquad ...(2)$$

$$f \in K \text{ iff for some } \phi \in S_0^* \; \frac{zf'(z)}{\phi(z)} = p(z).\; z \in E,\; p \in P \qquad \ldots(3)$$

$$f \in V_k \text{ iff } zf'(z) = \frac{(S_1(z)/z)^{1/2(k+2)/2}}{(S_2(z)/z)^{1/2(k-2)/2}} \qquad \ldots(4)$$

$S_1, S_2 \in S_0^*$.

The subclasses S_α^*, C_α, K and V_k are respectively, classes of functions starlike of order α, convex of order α, closed-to-convex and of bounded boundary rotation at most $k\pi$. We shall denote the subclasses S_0^* and C_0 respectively, by S^* and C. From (1) and (2) it is clear that

$$f \in C_\alpha \text{ iff } zf'(z) \in S_\alpha^*,\; \alpha < 1,\; z \in E. \qquad \ldots(5)$$

S is the extensively studied class of regular univalent functions in E normalized so that $f(0) = f'(0) - 1 = 0$. It is well known that for $0 \leqslant \alpha < 1$, $S_\alpha^* \in S$, $-\frac{1}{2} \leqslant \alpha < 1$, $C_\alpha \in S$, $K \in S$ and for $2 \leqslant k \leqslant 4$, $V_k \in S$. However, outside these ranges of the parameters α and k mentioned above, C_α and V_k are only locally univalent. Further, $V_2 = C$.

For a locally univalent function f in E the curvature $K_r^f(z)$ at the point $w = f(z)$ of the level line, i.e. the image of the circle $|z| = r$ under the mapping f, is given by

$$K_r^f(z) = \frac{1}{|zf'(z)|} \operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\}. \qquad \ldots(6)$$

Let $\underline{K}_r^B$ and $\overline{K}_r^B$ denote respectively, the infimum and supremum of $K_r^f(z)$ for $|z| = r$ when f belongs to a certain subclass B, of locally univalent functions in A, which is normal and compact. Zmorovic (1965) had obtained precise values of $\underline{K}_r^{S_\alpha^*}$ and $\overline{K}_r^{S_\alpha^*}$ for $0 \leqslant \alpha < 1$ and Korickii (1955, 1960) got precise bounds for certain subclasses of locally univalent functions. For $0 \leqslant \alpha < 1$ the exact values of $\underline{K}_r^{C_\alpha}$ and $\overline{K}^{C_\alpha}$ had been obtained by Zderkiewicz (1973) whereas for $\alpha = 0$ the sharp values had been obtained by Zmorovic (1952), Keogh (1954) had obtained the value of $\underline{K}_r^C$, and Eenigenburg (1970) had obtained the value of the infinimum of $K_r^f(z)$ for f belonging to a subclass of C_α. For the class V_k Noonan (1973) had obtained the precise bounds. But to the best of our knowledge the precise bounds $\underline{K}_r^{C_\alpha}$ and $\bar{K}_r^{C_\alpha}$ for $\alpha < 1$, where α is allowed to be negative and also the values of $\underline{K}_r^K$ and $\bar{K}_r^K$ are not yet known. Further, the value of $\underline{K}_r^S$ and bounds for some other classes of univalent functions are known (Mirosnicenko 1951, 1965) but the value of $\bar{K}_r^S$ is not yet known. Since for $-\frac{1}{2} \leqslant \alpha < 0$ the functions of the class C^α are univalent it is of interest to extend these results to negative values of α. In the present paper we give a method by which one can obtain precise values of $\underline{K}_r^B$, $\bar{K}_r^f$ when B is any of the subclasses defined above. Our method, moreover, is different from the methods used earlier and depends upon an inequality (stated in Theorem IV), for starlike function, which is of interest in itself.

We prove the following:

Theorem I — If $f \in C_\alpha$, $\alpha < 1$, is the class of functions defined by (2) and $\underline{K}_r^{C_\alpha}$ and $\bar{K}_r^{C_\alpha}$ are defined above, then

$$\underline{K}_r^{C_\alpha} = \frac{(1-r^2)^{1-\alpha}}{r} \text{ for } 0 \leqslant \alpha \leqslant 1 \qquad \ldots(7)$$

$$= \left(\frac{1+r}{r}\right)^{1-2\alpha} (1 + (2\alpha - 1)\, r),\ \alpha \leqslant 0 \qquad \ldots(8)$$

and

$$\bar{K}_r^{C_\alpha} = \frac{2e^{-1}}{(1-r^2)^\alpha \log \dfrac{1+r}{1-r}} \left[\frac{1+r}{1-r}\right]^{(1+r^2-2\alpha r^2)/2r}$$

$$\text{for } \alpha \leqslant \frac{1+r}{2r} - \frac{1}{(1+r)\log \dfrac{1+r}{1-r}} \qquad \ldots(9)$$

$$= \frac{1+r(1-2\alpha)}{r(1-r)^{2\alpha-1}} \text{ for } \frac{1+r}{2r} - \frac{1}{(1+r)\log \dfrac{1+r}{1-r}} \leqslant \alpha \leqslant 1. \qquad \ldots(10)$$

The above inequalities are sharp and the extremal functions are given below where equality, in each case, is attained at $z = r$:

(i) For equality in (7)

$$zf'(z) = \frac{z}{[(1-ze^{i\theta})^\lambda (1-ze^{-i\theta})^{1-\lambda}]^{2(1-\alpha)}},\ 0 \leqslant \lambda \leqslant 1,\ 0 \leqslant \alpha < 1 \qquad \ldots(11)$$

and $\cos\theta = r$.

(ii) For equality in (8)

$$zf'(z) = \frac{z}{(1+z)^{2(1-\alpha)}}, \; \alpha \leqslant 0. \qquad ...(12)$$

(iii) For equality in (9)

$$zf'(z) = \frac{z}{[(1-z)^{1+\lambda}(1+z)^{1-\lambda}]^{(1-\alpha)}}, \; -1 \leqslant \lambda \leqslant 1 \qquad ...(13)$$

where λ satisfies

$$\lambda(1-\alpha) \log \frac{1+r}{1-r} = 1 + \left(\alpha r - \frac{1+r^2}{2r}\right) \log \frac{1+r}{1-r}. \qquad ...(14)$$

(iv) For equality in (10)

$$zf'(z) = \frac{z}{(1-z)^{2(1-\alpha)}}. \qquad ...(15)$$

Theorem II — If $f \in K$ is the class of functions defined by (3) and $\underline{K}_r^K$ and $\bar{K}_r^K$ are defined above, then

$$\underline{K}_r^K = \frac{(1+r)^2}{r(1-r)^2}(1+r^2-4r) \qquad ...(16)$$

and

$$\bar{K}_r^K = \frac{4r}{e^2(1-r^2)}\left(\frac{1+r}{1-r}\right)^{(1+r^2)/r} \frac{1}{\left[\log \frac{1+r}{1-r}\right]^2} \qquad ...(17)$$

if $\qquad \frac{r}{1-r+r^2} \geqslant \frac{1}{2} \log \frac{1+r}{1-r}$

$$= \frac{(1-r+r^2)^2}{r(1-r)^2} \text{ if } \frac{r}{1-r+r^2} \leqslant \frac{1}{2} \log \frac{1+r}{1-r}. \qquad ...(18)$$

The above inequalities are sharp and the extremal functions are given below where equality, in each case, is attained at $z = r$:

(i) For equality in (16)

$$zf'(z) = \frac{z(1-z)}{(1+z)^3}. \qquad ...(19)$$

(ii) For equality in (17)

$$zf'(z) = \left(\frac{1+z}{1-z}\right)^{\lambda} \frac{1}{2} \log \frac{1+z}{1-z}, \; -1 \leqslant \lambda \leqslant 1 \qquad ...(20)$$

where λ satisfies

$$\lambda \log \frac{1+r}{1-r} = 2 - \frac{1+r^2}{r} \log \frac{1+r}{1-r}.$$

(iii) For equality in (18)

$$zf'(z) = \frac{1-z}{1+z} \frac{z}{1-z+z^2}. \qquad ...(21)$$

Theorem III — If $f \in V_k (k \geqslant 2)$ is the class of functions defined by (4) and $\underline{K}_r^{V_k}$ and $\bar{K}_r^{V_k}$ are defined above, then

$$\underline{K}_r^{V_k} = \frac{1}{r} \left(\frac{1+r}{1-r}\right)^{k/2} (1 - kr + r^2) \qquad ...(22)$$

and

$$\bar{K}_r^{V_k} = \frac{1-r^2}{r} \left[e(1-r^2)\left(\frac{1-r}{1+r}\right)^{(1+r^2)/2r} \frac{1}{2r} \log \frac{1+r}{1-r}\right]^{-(k+2)/4}. \qquad ...(23)$$

The above inequalities are sharp and the extremal functions are given below where equality in each case, is attained at $z = r$:

(i) For equality in (22)

$$f'(z) = \frac{1}{(1-z^2)} \left(\frac{1-z}{1+z}\right)^{k/2}. \qquad ...(24)$$

(ii) For equality in (23)

$$f'(z) = \left[\frac{1}{(1-z^2)} \left(\frac{1+z}{1-z}\right)^{\lambda}\right]^{(k+1)/2} (1 + ze^{i\theta})^{(k-2)/4}, \; -1 \leqslant \lambda \leqslant 1 \qquad ...(25)$$

where

$$\lambda \log \frac{1+r}{1-r} = 1 - \frac{1+r^2}{2r} \log \frac{1+r}{1-r} \qquad ...(26)$$

and

$$\cos\theta = \frac{1}{2r}\left[\frac{(1-r^2)^2}{r} \log \frac{1+r}{1-r} - (1+r^2)\right]. \qquad ...(27)$$

Theorem IV — If $f \in S^*$ is defined by (2) for $\alpha = 0$, $z \in E$, $|z| = r$, then

$$(1-r^2)\left|\frac{f(z)}{z}\right| \leqslant \text{Re}\, z\frac{f'(z)}{f(z)} \leqslant \frac{1+r}{1-r} + \frac{2r \log\left|(1-r)^2 \frac{f(z)}{z}\right|}{(1-r^2) \log \frac{1+r}{1-r}}. \qquad ...(28)$$

Both sides of the above inequality are sharp. For the left-hand inequality equality is attained at $z = r$ for functions of the form

$$f(z) = \frac{z}{(1 - ze^{i\gamma})^{2\lambda}(1 - ze^{-i\gamma})^{2(1-\lambda)}},\ 0 \leqslant \lambda \leqslant 1,\ \gamma \text{ real} \qquad \ldots(29)$$

and for the right-hand inequality, the extremal function is given by

$$f(z) = \frac{z}{(1 - z)^{2\lambda}(1 + z)^{2(1-\lambda)}},\ 0 \leqslant \lambda \leqslant 1 \qquad \ldots(30)$$

where λ satisfies

$$2\lambda \log \frac{1 + r}{1 - r} = \log \left[(1 + r)^2 \left| \frac{f(z)}{z} \right| \right] \qquad \ldots(31)$$

the equality being attained for $z = r$.

Corollary 1 — If $f \in S^*_\alpha$, $z \in E$, $z = r$, $\alpha < 1$, then

$$\alpha + (1 - \alpha)(1 - r^2)\left|\frac{f(z)}{z}\right|^{1/(1-\alpha)} \leqslant \operatorname{Re} \frac{zf'(z)}{f(z)}$$

$$\leqslant \frac{1 + (1 - 2\alpha) r}{1 - r} + \frac{2r \log \left| (1 - r)^{2(1-\alpha)} \dfrac{f(z)}{z} \right|}{(1 - r^2) \log \dfrac{1 + r}{1 - r}}. \qquad \ldots(32)$$

Theorem IV, besides being a useful tool in deriving bounds on curvature, is of considerable interest in itself because it gives bounds on $\operatorname{Re} zf'(z)/f(z)$ when $|f(z)|$ is fixed and $f \in S^*$ and is to be compared with the following inequality of Twomey (1970)

$$\left|\frac{zf'(z)}{f(z)}\right| \leqslant 1 + \frac{r \log \left| (1 + r)^2 \dfrac{f(z)}{z} \right|}{(1 - r) \log \dfrac{1 + r}{1 - r}},\quad |z| = r < 1,\ f \in S^*. \qquad \ldots(33)$$

In fact, the region of variability of $\log \frac{f(z)}{z}$ for fixed $z\frac{f'(z)}{f(z)}$, $f \in S^*$, $z \in E$ is not yet known though it is easy to see that this is a convex region and the inequalities (28) and (33) give partial solution of this problem. It may also be remarked that the left-hand inequality in (28) can be obtained from the Herglotz (1911) formula for functions of the class P and the right-hand inequality is derivable from the Hummel (1958) variation method.

Our method of derivation of (28) is quite elementary and shows that the region of variability of $\log \frac{f(z)}{z}$ for fixed $z\frac{f'(z)}{f(z)}$, $f \in S^*$ lies within a circle whose centre and radius are explicit functions of $z\,\frac{f'(z)}{f(z)}$ and $|z|$. Further, we also obtain the following:

Theorem V — If $p \in P$, $z \in E$, $|z| = r$, then

$$\int_0^1 \frac{(1-t)(1-tr^2) + t(1-r^2)\,\mathrm{Re}\,p(z)}{1-r^2t^2}\,dt \leqslant \mathrm{Re}\int_0^1 p(tz)\,dt$$

$$\leqslant \int_0^1 \frac{(1-t^2r^2)\,\mathrm{Re}\,p(z)\,dt}{t(1-r^2) + (1-t)(1-tr^2)\,\mathrm{Re}\,p(z)}. \qquad \ldots(34)$$

These inequalities are an improvement on the results of Robertson (1964) whose technique can hardly be adopted when $\mathrm{Re}\,p(z)$ is fixed. Finally, we mention that our method also provides us with a tool to obtain the radius of starlikeness of that subclass of A for which $f(0) = f'(0) - 1 = 0$ and $\mathrm{Re}\,f'(z) > 0$, $z \in E$. But the expressions involved are rather complicated and we only mention that it has been shown (Singh and Singh 1977) that this is greater than 0.8534.

2. Proofs of Theorems IV and V

We need the following:

Lemma 1 — If $z \in E$, $|z| = r$, $0 \leqslant t \leqslant 1$ and $p \in P$, then

$$\left| p(tz) + \frac{At - 2(1-t^2r^2)[(1-t^2r^2)\,\mathrm{Re}\,p(z) + it(1-r^2)\,\mathrm{Im}\,p(z)]}{B} \right| \leqslant \frac{At}{B} \qquad \ldots(35)$$

where

$$A = (1-r^2)(1-t)(1-tr^2)(\rho^2 - |p(z) - a|^2)$$

$$B = 2(1-t^2r^2)[t(1-r^2) + (1-t)(1-tr^2)\,\mathrm{Re}\,p(z)]$$

and

$$a = \frac{1+r^2}{1-r^2},\ \rho = \frac{2r}{1-r^2}.$$

For $p \in P$, $z, \xi \in E$, the function

$$F(z, \xi) = \frac{p(z) - p(\xi)}{p(z) + \overline{p(\xi)}}\,\frac{1 - \bar{\xi}z}{z - \xi} \qquad \ldots(36)$$

is analytic in E and satisfies $|F(z, \xi)| < 1$. Hence, the function

$$\phi(z, \xi) = \frac{F(z, \xi) - F(0, \xi)}{1 - \overline{F(0, \xi)}\,F(z, \xi)} \qquad \ldots(37)$$

satisfies the conditions of Schwarz lemma. On using

$$| \phi(z, \xi) | \leqslant | z | \qquad ...(38)$$

putting $z = t\xi$ and changing ξ into z we obtain the lemma after some elementary calculations.

The expression (35) can also be put in the following form:

$$\left| \frac{p(tz) - 1}{t} + \frac{A + 2(1 - t^2 r^2)[1 - r^2 - (1 + r^2) \operatorname{Re} p(z) + 2tr^2 \operatorname{Re} p(z) - i(1 - r^2) \operatorname{Im} p(z)]}{B} \right| \leqslant \frac{A}{B}. \qquad ...(39)$$

Lemma 2 — If $f \in S^*$, $z \in E$, $| z | = r$, $p(z) = zf'(z)/f(z)$, then

$$\left| \log \left\{ \frac{(1 - r^2) f(z)}{(\operatorname{Re} p(z)) z} \right\} + \int_0^1 \frac{A}{B} dt - i \operatorname{Im} p(z) \int_0^1 \frac{(1 - r^2) dt}{t(1 - r^2) + (1 - t)(1 - tr^2) \operatorname{Re} p(z)} \right|$$

$$\leqslant \int_0^1 \frac{A}{B} dt. \qquad ...(40)$$

It is enough to note that if

$$z \frac{f'(z)}{f(z)} = p(z)$$

then

$$\log \frac{f(z)}{z} = \int_0^1 \frac{p(tz) - 1}{t} dt.$$

On integrating the expression, inside the modulus in the left-hand side of (39) we obtain (40).

Proof of Theorem IV

Taking the real part of the expression under the modulus sign in (40) we obtain

$$\left| \log \left| \frac{(1 - r^2) f(z)}{(\operatorname{Re} p(z)) z!} \right| + \int_0^1 \frac{A}{B} dt \right| \leqslant \int_0^1 \frac{A}{B} dt.$$

Hence we have

$$\log \left| \frac{(1 - r^2)}{\operatorname{Re} p(z)} \frac{f(z)}{z} \right| \leqslant 0$$

which gives the left-hand inequality of (28). Further

$$- \log \left| \frac{(1 - r^2)}{\operatorname{Re} p(z)} \frac{f(z)}{z} \right| \leqslant \int_0^1 \frac{2A}{B}\, dt. \qquad \ldots(41)$$

The integral on the right-hand side can be evaluated explicitly and gives

$$\int_0^1 \frac{2A}{B}\, dt = \frac{2a \operatorname{Re} p(z) - 1 - |p(z)|^2}{2a \operatorname{Re} p(z) - 1 - (\operatorname{Re} p(z))^2} \Big[\{(1 + r^2) - (1 - r^2) \operatorname{Re} p(z)\} \times \frac{1}{2r} \log \frac{1 + r}{1 - r} + \log \operatorname{Re} p(z) \Big]$$

$$\leqslant \{(1 + r^2) - (1 - r^2) \operatorname{Re} p(z)\} \frac{1}{2r} \log \frac{1 + r}{1 - r} + \log \operatorname{Re} p(z). \qquad \ldots(42)$$

Combining (41) and (42) we obtain the right-hand inequality of (28).

The case of equality can be directly verified. One only needs to notice that on account of freedom of rotation we may take $z = r$ and further that equality will hold only when $p(r)$ is real.

In order to prove corollary 1 we observe that $f \in S^*$ iff $z\left(\frac{f(z)}{z}\right)^{1-\alpha} \in S^*_\alpha$, $\alpha < 1$.

We remark that the right-hand inequality of (28) can also be written in the form:

$$\operatorname{Re} \frac{zf'(z)}{f(z)} \leqslant \frac{1 + r^2}{1 - r^2} + \frac{2r}{1 - r^2} \frac{\log \left| (1 - r^2) \frac{f(z)}{z} \right|}{\log \frac{1 + r}{1 - r}}. \qquad \ldots(43)$$

which gives the following:

Corollary 2 — If $f \in S^*$ in E and $| f(z) | \leqslant \frac{r}{1 - r^2}$, then

$$\operatorname{Re} \frac{zf'(z)}{f(z)} \leqslant \frac{1 + r^2}{1 - r^2}. \qquad \ldots(44)$$

The left-hand inequality of (21) yields

Corollary 3 — If $f \in S^*$ in E, then

$$| f'(z) | > \left| \frac{f(z)}{z} \right|^2 (1 - r^2). \qquad \ldots(45)$$

Proof of Theorem V

If we integrate the expression under the modulus sign in (35) and then take its real part the theorem follows by argument similar to that used for proof of Theorem III.

We note that if $p \in P$, and

$$g(z) = \frac{1}{z}\int_0^z p(\xi)\,d\xi = \int_0^1 p(tz)\,dt$$

then $g \in P$. Robertson (1964) had shown that such functions g are subordinate to $\int_0^1 \frac{1-tz}{1+tz}\,dt$. In our case Theorem V gives the bounds on Re $g(z)$ when Re $p(z)$ is fixed. The method of Robertson does not apply to such situations. We further notice that both sides of the inequality (34) are monotone functions of Re $p(z)$ and hence on taking extreme values we have

$$-1 + \frac{2}{r}\log(1+r) \leqslant \operatorname{Re}\int_0^1 p(tz)\,dt \leqslant -1 + \frac{2}{r}\log\frac{1}{1-r} \quad \ldots(46)$$

which had been established by Robertson (1964).

3. Proof of Theorem I

In view of (4) if $f \in C_\alpha$, $zf'(z) = \phi(z) \in S_\alpha^*$ and therefore

$$K_r^f = \frac{1}{|zf'(z)|}\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} = \frac{1}{|\phi(z)|}\operatorname{Re}\frac{z\phi'(z)}{\phi(z)},\ \phi \in S_\alpha^*.$$

Hence

$$\underline{K}_r^{C_\alpha} = \min_{\phi \in S_\alpha^*} \frac{1}{|\phi(z)|}\operatorname{Re}\frac{z\phi'(z)}{\phi(z)}.$$

In order to obtain this minimum, in view of (32), we need to find minimum of the expression

$$F(t) = \frac{1}{r}\left[\alpha t^{-1} + (1-\alpha)(1-r^2)\, t^{\alpha/(1-\alpha)}\right],\ t = \left|\frac{\phi(z)}{z}\right|. \quad \ldots(47)$$

It is easily seen that for $\alpha > 0$ the minimum is attained for

$$t = (1-r^2)^{-(1-\alpha)}$$

and because this value of t lies within the range of variation of t this gives the minimum. We thus obtain (7). If $\alpha < 0$ then the minimum is readily seen to be attained where t is minimum, i.e.

$$t = (1 + r)^{2(1-\alpha)}$$

and this yields (8). The cases of equality for (7) and (8) can be ascertained by direct computation.

In like manner we see that

$$\bar{K}_r^{C_\alpha} = \min_{\phi \in S_\alpha^*} \frac{1}{|\phi(z)|} \operatorname{Re} \frac{z\phi'(z)}{\phi(z)}$$

and we need to find the maximum of

$$F(t) = \frac{1}{rt} [\log t + A_1] \frac{2r}{(1 - r^2) \log \frac{1+r}{1-r}} \qquad \text{...(48)}$$

where

$$A_1 = \frac{(1 + r)(1 + (1 - 2\alpha) r)}{2r} \log (1 + r)$$

$$- \frac{(1 - r)(1 - (1 - 2\alpha) r)}{2r} \log (1 - r)$$

and $\qquad t = \left| \dfrac{\phi(z)}{z} \right|$ lies in the interval

$$\left[\frac{1}{(1 + r)^{2(1-\alpha)}}, \frac{1}{(1 - r)^{2(1-\alpha)}}\right]. \qquad \text{...(49)}$$

It is readily confirmed that the maximum of $F(t)$ is attained for $t = e^{1-A_1}$ which is the root of $F'(t) = 0$.

For this value of t the value of $F(t)$ is seen to be given by (9). This establishes (9) except for the fact that we still need to ascertain that t lies in the interval given by (49). Equivalently we need to find if the following holds for $0 < r < 1$

$$\frac{(1 - r)(1 - (1 - 2\alpha) r)}{2r} \log \frac{1 + r}{1 - r} \leqslant 1$$

$$\leqslant \frac{(1 + r)(1 + (1 - 2\alpha) r)}{2r} \log \frac{1 + r}{1 - r}. \qquad \text{...(50)}$$

The left-hand inequality is always satisfied but the right-hand inequality is satisfied only if

$$\alpha \leqslant \frac{1+r}{2r} - \frac{1}{(1+r)\log\frac{1+r}{1-r}}. \qquad \ldots(51)$$

However, if

$$\frac{1+r}{2r} - \frac{1}{(1+r)\log\frac{1+r}{1-r}} \leqslant \alpha \leqslant 1 \qquad \ldots(52)$$

one easily confirms that $F(t)$ is a monotone increasing function of t and hence its maximum is attained for $t = \frac{1}{(1-r)^{2(1-\alpha)}}$. This gives (10). The case of equality is confirmed by direct computation.

4. Proof of Theorem II

In view of (3) we have to find the extreme values of

$$\frac{1}{|\,p(z)f(z)\,|} \operatorname{Re}\left\{z\frac{p'(z)}{p(z)} + \frac{zf'(z)}{f(z)}\right\},\ p \in P,\ f \in S^*. \qquad \ldots(53)$$

We need the following :

Lemma 3 — If $p \in P$, then

$$|\,p(z)\,| - \frac{1+r^2}{1-r^2} \leqslant \operatorname{Re} z\frac{p'(z)}{p(z)} \leqslant \frac{1+r^2}{1-r^2} - \frac{1}{|\,p(z)\,|},\ |\,z\,| = r. \ldots(54)$$

As $p(z)$ and $zp'(z)$ satisfy the following well-known inequality

$$|\,zp'(z) - \tfrac{1}{2}(p^2(z)-1)\,| \leqslant \frac{\rho^2 - |\,p(z)-a\,|^2}{2} \qquad \ldots(55)$$

where a and ρ are defined in Lemma 1, we obtain for $p(z) = |\,p(z)\,|\,e^{i\theta}$

$$\left[\tfrac{1}{2}\left(|\,p(z)\,| - \frac{1}{|\,p(z)\,|}\right) - a\right]\cos\theta + \frac{1+|\,p(z)\,|^2}{2\,|\,p(z)\,|} \leqslant 2\operatorname{Re}\frac{zp'(z)}{p(z)}$$

$$\leqslant \left[\tfrac{1}{2}\left(|\,p(z)\,| - \frac{1}{|\,p(z)\,|}\right) + a\right]\cos\theta - \frac{1+|\,p(z)\,|^2}{2\,|\,p(z)\,|}. \qquad \ldots(56)$$

It is easily confirmed that the minimum of the left-hand side and the maximum of right-hand side are attained for $\theta = 0$ and this gives us (54).

Proof of Theorem II — Taking into consideration (53), (54) and (28) and using the notation $|\,p(z)\,| = x$, $|\,f(z)\,| = y$, we find that for

$$\frac{1-r}{1+r} \leqslant x \leqslant \frac{1+r}{-r} \qquad \ldots(57)$$

and

$$\frac{r}{(1+r)^2} \leqslant y \leqslant \frac{r}{(1-r)^2} \qquad \text{...(58)}$$

we need to obtain the minimum of

$$F(x, y) = \frac{1}{xy}\left[x - \frac{1+r^2}{1-r^2} + \frac{1-r^2}{r}\, y\right] \qquad \text{...(59)}$$

and maximum of

$$F_1(x, y) = \frac{1}{xy}\left[\frac{1+r^2}{1-r^2} - \frac{1}{x} + \frac{1+r}{1-r} + \frac{2r}{1-r^2}\, \frac{\log\left\{\frac{(1-r^2)}{r}\, y\right\}}{\log \frac{1+r}{1-r}}\right]. \qquad \text{...(60)}$$

This reduces the problem to finding extreme values of functions of two real variables. It is readily verified in the case of (59) that the roots of

$$\frac{\partial F}{\partial x}(x, y) = 0 = \frac{\partial F}{\partial y}(x, y)$$

do not give the minimum and that the minimum is attained for $x = \frac{1-r}{1+r}$ and $y = \frac{r}{(1+r)^2}$ and this value is given by (16).

The case of equality is confirmed by direct calculation.

In order to maximize $F_1(x, y)$ given by (60) it is found that the equations

$$\frac{\partial F_1}{\partial x} = 0 = \frac{\partial F_1}{\partial y}$$

give

$$x = \frac{1-r^2}{2r} \log \frac{1+r}{1-r} \qquad \text{...(61)}$$

and

$$r \log\left(\frac{(1-r)^2}{r}\, y\right) = 2r - (1 + r + r^2) \log \frac{1+r}{1-r}. \qquad \text{...(62)}$$

The value of x given by (61) satisfies (57) because

$$\frac{r}{(1+r)^2} \leqslant \tfrac{1}{2} \log \frac{1+r}{1-r} \leqslant \frac{r}{(1-r)^2}$$

and the value of y given by (62) satisfies the right-hand inequality of (58) because

$$\frac{r}{1+r+r^2} \leqslant \tfrac{1}{2}\log\frac{1+r}{1-r}$$

but it satisfies the left-hand inequality of (58) if

$$\frac{r}{1-r+r^2} \geqslant \tfrac{1}{2}\log\frac{1+r}{1-r}. \qquad \ldots(63)$$

The inequality (63) does not always hold and in fact the equation

$$\frac{r}{1-r+r^2} = \tfrac{1}{2}\log\frac{1+r}{1-r}$$

has exactly one root r_0 such that $\frac{1}{2} < r_0 < 1$. This proves (17). When (63) does not hold the maximum value of $F_1(x, y)$ is attained for $x = \dfrac{1-r^2}{1-r+r^2}$ and $y = \dfrac{r}{(1+r)^2}$ and the corresponding value of $\bar{K}_r^K$ is given by (18). The case of equality can be directly confirmed.

5. Proof of Theorem III

Let us put

$$k_1 = \tfrac{1}{2}\left(\frac{k}{2}-1\right), \left|\frac{S_1(z)}{z}\right| = u \text{ and } \left|\frac{S_2(z)}{z}\right| = v \qquad \ldots(64)$$

where u and v lie in the interval

$$\left[\frac{1}{(1+r)^2}, \frac{1}{(1-r)^2}\right]. \qquad \ldots(65)$$

In view of (4) we need to find the extreme values of

$$K_r^f(z) = \frac{1}{r}\frac{v^{k_1}}{u^{k_1+1}}\operatorname{Re}\left[(k_1+1)\,z\frac{S_1'(z)}{S_1(z)} - k_1 z\frac{S_2'(z)}{S_2(z)}\right]. \qquad \ldots(66)$$

In view of (28) and (66) we need to obtain the minimum of

$$F_2(u, v) = \frac{1}{r}\frac{v^{k_1}}{u^{k_1+1}}\left[(k_1+1)(1-r^2)u - k_1\left\{\frac{1+r}{1-r} + \frac{2r\log[(1-r)^2u]}{(1-r^2)\log\frac{1+r}{1-r}}\right\}\right] \qquad \ldots(67)$$

and the maximum of

$$F_3(u, v) = \frac{1}{r} \frac{v^{k_1}}{u^{k_1+1}}$$

$$\times \left[(k_1 + 1) \left\{ \frac{1+r}{1-r} + \frac{2r \log [(1-r)^2 u]}{(1-r^2) \log \frac{1+r}{1-r}} \right\} - k_1(1-r^2) v \right] \quad \ldots(68)$$

when u and v lie in the interval given by (65).

It is readily confirmed that the equations

$$\frac{\partial}{\partial u} F_2(u, v) = 0 = \frac{\partial}{\partial v} F_2(u, v)$$

do not give the minimum. Hence the minimum is attained on the boundary for $u = \frac{1}{(1+r)^2}$ and $v = \frac{1}{(1-r)^2}$. This yields (22). The case of equality is easy to check for the function given by (24).

Further, the equations

$$\frac{\partial}{\partial u} F_3(u, v) = 0 = \frac{\partial}{\partial v} F_3(u, v)$$

give that maximum is attained for

$$u = e(1-r)^{(1-r)^2/2r} (1+r)^{(1+r)^2/2r} \quad \ldots(69)$$

and

$$v = \frac{2r}{(1-r^2)^2 \log \frac{1+r}{1-r}}. \quad \ldots(70)$$

This gives (23) and simple calculation confirms the case of equality for the function given by (25).

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