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ON AN EXTENSION OF A THEOREM OF LJUBOMIR CIRIC

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The aim of this note is to extend a theorem of Ciric (1977) for a pair of mappings.

In a recent paper Ciric (1977) proved the following theorem:

Theorem C1 — Let X be a closed convex subset of a normed space and let $T : X \to X$ satisfy the condition:

$$d(Tx, Ty) \leqslant q \max \{cd(x, y), [d(x, Tx) + d(y, Ty)], [d(x, Ty) + d(y, Tx)]\}$$

where $c \geqslant 0$, $0 \leqslant q < 1$. If a sequence

$$x_{n+1} = (1 - t) x_n + t Tx_n, n = 0, 1, 2, ..., x_0 \in X, 0 < t < 1$$

converges in X then T has a fixed point.

The aim of this note is to extend Theorem C1 for a pair of mappings.

We have the following result :

Theorem 1 — Let X be a closed convex subset of a normed space and let T_1 and T_2 be two mappings of X into itself such that

$$d(T_1x, T_2y) \leqslant q \max \{cd(x, y), [d(x, T_1x) + d(y, T_2y)], [d(x, T_2y) + d(y, T_1x)]\} \quad ...(1)$$

where $c \geqslant 0$, $0 \leqslant q < 1$. If a sequence $\{x_n\}$ where

$$x_{2n+1} = (1 - t) x_{2n} + t T_1x_{2n},$$
$$x_{2(n+1)} = (1 - t) x_{2n+1} + t T_2x_{2n+1}, n = 0, 1, 2, ..., \quad x_0 \in X, 0 < t < 1,$$

converges in X then T_1 and T_2 have a common fixed point.

PROOF : Let $z \in X$ such that $\lim x_n = z$.

We shall show that z is a fixed point of T_1 and T_2.

$$d(z, T_2z) \leqslant d(z, x_{2n+1}) + \| (1 - t) x_{2n} + t T_1x_{2n} - T_2z \|$$
$$= d(z, x_{2n+1}) + \| (1 - t) x_{2n} - (1 - t) T_2z + t T_1x_{2n} - t T_2z \|$$
$$\leqslant d(z, x_{2n+1}) + (1 - t) d(x_{2n}, T_2z) + t d(T_1x_{2n}, T_2z)$$

i.e.

$$d(T_2z, z) \leqslant d(z, x_{2n+1}) + (1 - t)\, d(x_{2n}, T_2z)$$

$$+ qt \max \{cd(x_{2n}, z), [d(x_{2n}, T_1x_{2n}) + d(z, T_2z)],$$

$$[d(x_{2n}, T_2z) + d(z, T_1x_{2n})]\}$$

$$\leqslant d(z, x_{2n+1}) + (1 - t)\, d(x_{2n}, T_2z)$$

$$+ qt \max \{cd(x_{2n}, z), [d(x_{2n}, T_1x_{2n}) + d(z, T_2z)],$$

$$[d(x_{2n}, T_2z) + d(z, x_{2n}) + d(x_{2n}, T_1x_{2n})]\}$$

$$= d(z, x_{2n+1}) + (1 - t)\, d(x_{2n}, T_2z)$$

$$+ qt \max \left\{cd(x_{2n}, z), \left[\frac{1}{t}\, d(x_{2n}, x_{2n+1}) + d(z, T_2z)\right],\right.$$

$$\left.\left[d(x_{2n}, T_2z) + d(z, x_{2n}) + \frac{1}{t}\, d(x_{2n+1}\, x_{2n})\right]\right\}.$$

Letting $n \to \infty$, we obtain

$$d(z, T_2z) \leqslant (1 - t)\, d(z, T_2z) + qt\, d(z, T_2z)$$

which implies $d(z, T_2z) = 0$. Similarly one can show that $d(z, T_1z) = 0$. This completes the proof of the theorem.

Remarks : Theorem C1 follows from our theorem by taking

$$T_1 = T_2 = T.$$

Reference

Ciric, Ljubomir (1977). Quasi-contraction in Banach spaces. *Publ. De L' Inst. Math.*, Tome 21 (35), 41-48.