Indian J. pure appl. Math., **10**(2) : 147-157, February 1979

MELLIN TRANSFORM TECHNIQUE FOR SOLVING GRIFFITH CRACK PROBLEM AT THE INTERFACE OF TWO BONDED DISSIMILAR WEDGES

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(*Received* 25 *May* 1978)

The formulation of a general Griffith crack problem situated at the interface of two bonded dissimilar wedges is presented. A systematic use of Mellin transform reduces the problem to that of solving a system of simultaneous triple integral equations. This set of equations is shown to be equivalent to Riemann boundary value problem which has a closed form solution. As a specific example the problem of finding distribution of stresses for a pair of Griffith cracks situated at the interface of two bonded half planes is considered. Expressions for stress intensity factors have been derived.

1. INTRODUCTION

In composite materials, brittle fracture often takes place at the interface of dissimilar layers. To investigate such phenomena by the methods of fracture mechanics we must have at our disposal the solutions of the problems in the theory of elasticity for cracks between dissimilar bodies. Recently this type of problems have been extensively studied. Lowengrub and Sneddon (1973, 1974), Lowengrub (1975), Goldstein and Vainshelbaum (1976) and the authors (Srivastava *et al.* 1977) have shown that the method of integral transforms can be used for analysing this type of problems.

In this paper we present a general formulation of the problem of a Griffith crack situated at the interface of two wedges with different elastic properties bonded together. The problem is first reduced to a set of simultaneous triple integral equations by means of inverse Mellin transforms. This latter problem, in turn, is reduced to a Riemann boundary value problem. The closed form solution of Riemann boundary value problem is used for deriving expressions for the components of stresses and stress intensity factors at the tip of crack.

2. FORMULATION OF THE PROBLEM

A plane polar coordinate system is used. In this coordinate system the components of stress tensor are $[\sigma_{rr}(r, \theta), \sigma_{r\theta}(r, \theta), \sigma_{\theta\theta}(r, \theta)]$ and the components of displacement vectors are $[u_r(r, \theta), u_\theta(r, \theta)]$. We consider the plane strain problem of determining the stress field in the vicinity of a Griffith crack, described by $\theta = 0$, $a \leqslant r \leqslant b$, situated at the interface of two bonded dissimilar wedges. The upper wedge with elastic constants μ_1, η_1 is supposed to occupy the region $0 \leqslant r < \infty$, $0 \leqslant \theta < \pi/n$, and the lower wedge with elastic properties μ_2, η_2 occupies the region $0 \leqslant r < \infty$, $-\pi/n \leqslant \theta \leqslant 0$. Here μ is the modulus of rigidity and η is the Poisson's ratio. The subscripts 1 and 2 will be used to denote the upper and lower wedges respectively. If we assume that the upper and lower surfaces of crack are subjected to prescribed tractions, then inside the crack we have the conditions:

$$\sigma_{\theta\theta}(r, 0+) = \sigma_{\theta\theta}(r, 0-) = -p_1(r),\ a \leqslant r \leqslant b \qquad ...(2.1)$$

$$\sigma_{r\theta}(r, 0+) = \sigma_{r\theta}(r, 0-) = -p_2(r),\ a \leqslant r \leqslant b \qquad ...(2.2)$$

On the region outsides of the crack we have the following continuity conditions:

$$\left.\begin{aligned} \sigma_{\theta\theta}(r, 0+) &= \sigma_{\theta\theta}(r, 0-) \qquad &...(2.3)\\ \sigma_{r\theta}(r, 0+) &= \sigma_{r\theta}(r, 0-) \qquad &...(2.4)\\ u_r(r, 0+) &= u_r(r, 0-) \qquad &...(2.5)\\ u_\theta(r, 0+) &= u_\theta(r, 0-). \qquad &...(2.6) \end{aligned}\right\} (0 \leqslant r \leqslant a) \cup (b \leqslant r < \infty)$$

We further assume that at the free edges of the wedges the following conditions are satisfied

$$u_\theta(r, \pi/n) = u_\theta(r, -\pi/n) = 0,\ 0 \leqslant r < \infty \qquad ...(2.7)$$

$$\sigma_{r\theta}(r, \pi/n) = \sigma_{r\theta}(r, -\pi/n) = 0,\ 0 \leqslant r < \infty. \qquad ...(2.8)$$

3. Solution of the Equations of Elastic Equilibrium in Polar Coordinates

The solution of the equations of elastic equilibrium pertaining to plane strain in plane polar coordinates in terms of Airy's stress function as given in Sneddon (1963) is

$$(2\mu_i)^{-1} r^2 \sigma_{\theta\theta}(r, \theta) = M^{-1}[s(s+1)\,\bar{\chi}_i(s, \theta); r] \qquad ...(3.1)$$

$$(2\mu_i)^{-1} r^2 \sigma_{r\theta}(r, \theta) = M^{-1}\left[(s+1)\frac{\partial\bar{\chi}_i}{\partial\theta}(s, \theta); r\right] \qquad ...(3.2)$$

$$r u_r(r, \theta) = -M^{-1}\left[(s+1)^{-1}\left\{(1-\eta_i)\frac{\partial^2\bar{\chi}_i}{\partial\theta^2}(s, \theta) - (1+\eta_i s)\, s\bar{\chi}_i(s, \theta)\right\}; r\right] \qquad ...(3.3)$$

$$ru_\theta(r, \theta) = -M^{-1}\Big[\{(s+1)(s+2)\}^{-1}\Big((1-\eta_i)\frac{\partial^3\bar{\chi}_i}{\partial\theta^3}(s, \theta)$$

$$+ \{(1-\eta_i)s^2 + (s+1)(s+2)\}\frac{\partial\bar{\chi}_i}{\partial\theta}(s, \theta)\Big); r\Big] \quad ...(3.4)$$

where $\bar{\chi}_i(s, \theta) = A_i \sin(s\theta) + B_i \cos(s\theta) + C_i \sin(s+2)\theta + D_i \cos(s+2)\theta$

$$= \int_0^\infty r^{s-1}\chi_i(r, \theta)\,dr, \; i = 1, 2$$

and $M^{-1}[f(s); r]$ denotes the inverse Mellin transform of the function $f(s)$.

4. Some Useful Results

In this section we list some results which will be used in the next section. From the result

$$\int_a^b (x-a)^{m-1}(b-x)^{n-1}(a+x)^{-p}(b+x)^{-q}\,dx = (b-a)^{m+n-1}$$

$$\times (b+a)^{-p-q} F_3(m, n; p, q; m+n; -X, X), \; \mathrm{Re}\,(m, n) > 0 \quad ...(4.1)$$

where $X = (b-a)/(b+a)$ and F_3 is hypergeometric function of two variables defined in Erdelyi (1954a, p. 274) we can easily derive the following results. Let

$$I_n = \int_a^b x^n\{(a-x)(b+x)\}^{-(1/2)+i\omega}\{(a+x)(b-x)\}^{-(1/2)-i\omega}\,dx$$

then

$$I_0 = \frac{\pi}{(a+b)} \operatorname{sech} \pi\omega \; {}_2F_1(\tfrac{1}{2}+i\omega, \tfrac{1}{2}-i\omega; 1; X^2) \quad ...(4.2)$$

$$I_1 = \pi X \operatorname{sech}\pi\omega\,(\tfrac{1}{2}+i\omega)\,F_3(\tfrac{3}{2}+i\omega, \tfrac{1}{2}-i\omega; \tfrac{1}{2}-i\omega, \tfrac{1}{2}+i\omega; 2;$$

$$-X, X) + a\,I_0$$

$$= \pi X \operatorname{sech}\pi\omega\Big[(\tfrac{1}{2}+i\omega)\,F_3(\tfrac{1}{2}+i\omega, \tfrac{1}{2}-i\omega; \tfrac{1}{2}-i\omega, \tfrac{1}{2}+i\omega; 2;$$

$$-X, X) + \frac{X}{2}(\tfrac{1}{4}+\omega^2)\,F_3(\tfrac{3}{2}+i\omega, \tfrac{1}{2}-i\omega; \tfrac{3}{2}-i\omega, \tfrac{1}{2}+i\omega; 3;$$

$$-X, X)\Big] + a\,I_0$$

$$= I_1' + i\,I_1'' \quad ...(4.3)$$

$$I_2 = (b - a)\,\Gamma(\tfrac{3}{2} + i\omega)\,\Gamma(\tfrac{1}{2} - i\omega)\,F_3(\tfrac{3}{2} + i\omega, \tfrac{1}{2} - i\omega; \tfrac{1}{2} - i\omega, - \tfrac{1}{2} + i\omega; 2; - X, X) + a^2 I_0$$

$$= \pi(b - a)\,\mathrm{sech}\,\pi\omega\Big[(\tfrac{1}{2} + i\omega)\,F_3(\tfrac{1}{2} + i\omega, - \tfrac{1}{2} - i\omega; \tfrac{1}{2} + i\omega, - \tfrac{1}{2} + i\omega; 2; - X, X) - \frac{X}{2}(\tfrac{1}{4} + \omega^2)\{F_3(\tfrac{3}{2} + i\omega, - \tfrac{1}{2} - i\omega; \tfrac{3}{2} - i\omega, - \tfrac{1}{2} + i\omega; 3; - X, X) + F_3(\tfrac{1}{2} + i\omega, \tfrac{1}{2} - i\omega; \tfrac{1}{2} - i\omega, \tfrac{1}{2} + i\omega; 3; -X, X)\} + \frac{X^2}{6}(\tfrac{1}{4} + \omega^2)(\tfrac{1}{2} - i\omega)\,F_3(\tfrac{3}{2} + i\omega, \tfrac{1}{2} - i\omega; \tfrac{3}{2} - i\omega, \tfrac{1}{2} + i\omega; 4; - X, X)\Big] + a^2 I_0$$

$$= I'_2 + i\,I''_2. \qquad \ldots(4.4)$$

It is interesting to note that the above series are rapidly converging and for any numerical computation of I_n it is sufficient to consider only a few terms of the above series. We shall also use the following results given in (Erdelyi 1954b, p. 346)

$$M^{-1}[\tan(\pi s/n); r/u)] = - n(ru)^{n/2}/\pi(u^n - r^n), \quad 0 < s < n/2 \qquad \ldots(4.5)$$

$$M^{-1}[\cot(\pi s/n); r/u)] = nu^n/\pi(u^n - r^n), \quad 0 < s < n/2. \qquad \ldots(4.6)$$

5. Reduction of the Problem to Simultaneous Triple Integral Equations and their Solutions

We consider the two cases when $n = 1, 2$. The conditions (2.3), (2.4), (2.7) and (2.8) give the following equations

$$2\mu_1(B_1 + D_1) = 2\mu_2(B_2 + D_2) \qquad \ldots(5.1)$$

$$2\mu_1[A_1 s + C_1(s + 2)] = 2\mu_2[A_2 s + C_2(s + 2)] \qquad \ldots(5.2)$$

$$A_1 \cos(\pi s/n) - B_1 \sin(\pi s/n) = 0 = C_1 \cos(\pi s/n) - D_1 \sin(\pi s/n) \qquad \ldots(5.3)$$

$$A_2 \cos(\pi s/n) + B_2 \sin(\pi s/n) = 0 = C_2 \cos(\pi s/n) + D_2 \sin(\pi s/n) \qquad \ldots(5.4)$$

The conditions (2.5) and (2.6) lead to the following equations

$$r^{-1} M^{-1}[(s + 1)^{-1} \varphi(s); r] = 0, (0 \leqslant r \leqslant a) \cup (b \leqslant r < \infty) \qquad \ldots(5.5)$$

$$r^{-1} M^{-1}[(s + 1)^{-1} \psi(s); r] = 0, (0 \leqslant r \leqslant a) \cup (b \leqslant r < \infty) \qquad \ldots(5.6)$$

where

$$(A_1 - A_2)\, s + (C_1 - C_2)(s + 2) + 4\{(1 - \eta_2)\, C_2 - (1 - \eta_1)\, C_1\} = \varphi(s)/(s + 1) \quad ...(5.7)$$

$$(B_1 + D_1 - B_2 - D_2)\, s + 4\{(1 - \eta_1)\, D_1 - (1 - \eta_2)\, D_2\} = \psi(s)/(s + 1). \quad ...(5.8)$$

From these equations we get

$$-\frac{(\beta^2 - \alpha^2)}{2}(B_2 + D_2)\, s(s + 1) = \alpha\, \varphi(s) \cot(\pi\, s/n) + \beta\, \psi(s) \quad ...(5.9)$$

$$-\frac{(\beta^2 - \alpha^2)}{2}[A_2 s + C_2(s + 1)]\,(s + 1) = \alpha\, \psi(s) \tan(\pi\, s/n) + \beta\, \varphi(s) \quad ...(5.10)$$

where $\alpha = 4\,[G(1 - \eta_1) + (1 - \eta_2)]$, $\beta = 2\,[(1 - 2\eta_1)\, G - (1 - 2\eta_2)]$, $G = \mu_2/\mu_1$.

Hence we get

$$r^2\, \sigma_{\theta\theta}(r, 0+) = r^2\, \sigma_{\theta\theta}(r, 0-) = -M^{-1}\left[\frac{4\mu_2}{\beta^2 - \alpha^2}\left\{\alpha \cot\frac{\pi s}{n}\varphi(s) + \beta\, \psi(s)\right\}; r\right] \quad ...(5.11)$$

$$r^2\, \sigma_{r\theta}(r, 0+) = r^2\, \sigma_{r\theta}(r, 0-) = -M^{-1}\left[\frac{4\mu_2}{\beta^2 - \alpha^2}\left\{\alpha \tan\frac{\pi s}{n}\psi(s) + \beta\, \varphi(s)\right\}; r\right]. \quad ...(5.12)$$

By differentiating (5.5) and (5.6) we get

$$\left.\begin{aligned} M^{-1}[\varphi(s); r] &= 0 \\ M^{-1}[\psi(s); r] &= 0 \end{aligned}\right\} \qquad (0 \leqslant r \leqslant a) \cup (b \leqslant r < \infty).$$

These equations together with the conditions (2.1) and (2.2) lead to the following simultaneous triple integral equations involving inverse Mellin transforms

$$M^{-1}\left[\alpha\, \varphi(s) \cot\frac{\pi s}{n} + \beta\, \psi(s); r\right] = \frac{\beta^2 - \alpha^2}{4\,\mu_2}\, r^2 p_1(r) = f_1(r^{n/2}), \quad a \leqslant r \leqslant b \quad ...(5.13)$$

$$M^{-1}\left[\alpha\, \psi(s) \tan\frac{\pi s}{n} + \beta\varphi(s); r\right] = \frac{\beta^2 - \alpha^2}{4\,\mu_2}\, r^2 p_2(r) = f_2(r^{n/2}), \quad a \leqslant r \leqslant b \quad ...(5.14)$$

$$M^{-1}\,[\varphi(s); r] = 0, \qquad (0 \leqslant r \leqslant a) \cup (b \leqslant r < \infty) \quad ...(5.15)$$

$$M^{-1}[\psi(s); r] = 0, \qquad (0 \leqslant r \leqslant a) \cup (b \leqslant r < \infty) \qquad \ldots(5.16)$$

where $0 < s < n/2$. To solve the above set of equations we assume that

$$M^{-1}[\varphi(s); r] = \begin{cases} a_1(r^{n/2}), & a \leqslant r \leqslant b \\ 0, & (0 \leqslant r \leqslant a) \cup (b \leqslant r < \infty) \end{cases} \qquad \ldots(5.17)$$

$$M^{-1}[\psi(s); r] = \begin{cases} b_1(r^{n/2}), & a \leqslant r \leqslant b \\ 0, & (0 \leqslant r \leqslant a) \cup (b \leqslant r < \infty). \end{cases} \qquad \ldots(5.18)$$

Hence we have

$$\varphi(s) = \int_a^b a_1(u^{n/2})\, u^{s-1}\, du \qquad \ldots(5.19)$$

$$\psi(s) = \int_a^b b_1(u^{n/2})\, u^{s-1}\, du. \qquad \ldots(5.20)$$

Here $a_1(u)$ and $b_1(u)$ are unknown functions to be determined. By using (4.6) we have

$$M^{-1}\left[\varphi(s) \cot \frac{\pi s}{u}; r\right] = \int_a^b a_1(u^{n/2})\, M^{-1}\left[\cot \frac{\pi s}{n}; \frac{r}{u}\right] \frac{du}{u}$$

$$= \frac{n}{2\pi} \int_a^b a_1(u^{n/2})\, u^{(n/2)-1} [(u^{n/2} - r^{n/2})^{-1} + (u^{n/2} + r^{n/2})^{-1}]\, du$$

$$= \frac{1}{\pi} \int_A^B a_1(t) [(t - x)^{-1} + (t + x)^{-1}]\, dt$$

$$= \frac{1}{\pi} \int_L \frac{a(t)}{t - x}\, dt. \qquad \ldots(5.21)$$

Similarly from (4.5) we have

$$M^{-1}\left[\psi(s) \tan \frac{\pi s}{n}; r\right] = -\frac{n}{2\pi} \int_a^b b_1(u^{n/2})\, u^{(n/2)-1} [(u^{n/2} - r^{n/2})^{-1}$$

$$- (u^{n/2} + r^{n/2})^{-1}]\, du$$

$$= -\frac{1}{\pi} \int_A^B b_1(t) [(t - x)^{-1} - (t + x)^{-1}] dt$$

(*equation continued on p.* 153)

$$= -\frac{1}{\pi}\int_L \frac{b(t)}{t-x}\,dt$$

where $x = r^{n/2}$, $A = a^{n/2}$, $B = b^{n/2}$, $L = (-B, -A) \cup (A, B)$ and $a(t)$ and $b(t)$ are the extensions of $a_1(t)$ and $b_1(t)$ respectively to the region $(-B, -A)$ such that

$$\left.\begin{aligned} a(t) &= \begin{cases} a_1(t), & A \leqslant t \leqslant B \\ -a_1(-t), & -B \leqslant t \leqslant -A \end{cases} \\ b(t) &= \begin{cases} b_1(t), & A \leqslant t \leqslant B \\ b_1(-t), & -B \leqslant t \leqslant -A. \end{cases} \end{aligned}\right\} \qquad \ldots(5.22)$$

With the help of these equations (5.15) and (5.16) reduce to the following set of singular integral equations

$$\beta\, b(x) + \frac{\alpha}{\pi}\int_L \frac{a(t)}{t-x}\,dt = \hat{f}_1(x) \qquad \ldots(5.23)$$

$$\beta\, a(x) - \frac{\alpha}{\pi}\int_L \frac{b(t)}{t-x}\,dt = \hat{f}_2(x) \qquad \ldots(5.24)$$

where $\hat{f}_1(x)$ and $\hat{f}_2(x)$ are even and odd extensions of $f_1(x)$ and $f_2(x)$ respectively. The substitution

$$\lambda(x) = b(x) + ia(x) \qquad \ldots(5.25)$$

$$f(x) = \hat{f}_1(x) + i\hat{f}_2(x) \qquad \ldots(5.26)$$

helps in combining two eqns. (5.23) and (5.24) into a singular integral equation

$$\beta\,\lambda(x) + \frac{\alpha}{\pi i}\int_L \frac{\lambda(t)}{t-x}\,dt = f(x). \qquad \ldots(5.27)$$

This equation can be further reduced to Riemann boundary value problem (Muskhelishvili 1965) which has a closed form solution. If we define

$$\Lambda(z) = \frac{1}{2\pi i}\int_L \frac{\lambda(u)}{u-z}\,du, \ \mathrm{Re}\,(z) = x \qquad \ldots(5.28)$$

and use Pelmelj formula, we have

$$\Lambda^+(x) - \Lambda^-(x) = \lambda(x) \qquad \ldots(5.29)$$

$$\Lambda^+(x) + \Lambda^-(x) = \frac{1}{\pi i}\int_L \frac{\lambda(u)}{u-x}\,du \qquad \ldots(5.30)$$

These relations show that (5.27) is equivalent to the condition

$$\Lambda^+(x) = -k\,\Lambda^-(x) + (\alpha+\beta)^{-1} f(x),\ x \in L \qquad ...(5.31)$$

where $k = (\alpha - \beta)/(\alpha + \beta) > 0$. The solution of this problem as given in Muskhelishvili (1965) is

$$\Lambda(z) = \frac{X(z)}{2\pi i}\int_L \frac{f(t)}{X^+(t)\,(t-z)}\,dt + (Cz + D)\,X(z) \qquad ...(5.32)$$

where C and D are arbitrary complex constants and $X(z)$ is the solution of the homogeneous boundary value problem

$$X^+(t) = -k\,X^-(t), \quad t \in L. \qquad ...(5.33)$$

This problem has the solution

$$X(z) = \{(z - A)(z + B)\}^{-(1/2)+i\omega}\,\{(z + A)(z - B)\}^{-(1/2)-i\omega} \qquad ...(5.34)$$

where $\omega = \frac{1}{\pi}\log k$. In case $f(t)$ is a polynomial

$$\int_L \frac{f(t)}{X^+(t)\,(t-z)}\,dt = \frac{2\pi i}{k+1}\left[\frac{f(t)}{X(z)} - L(z)\right] \qquad ...(5.35)$$

where $L(z)$ is the term independent of 't' in the Laurant series expansion of the function $f(t)/X(t)\,(t - z)$. Hence we have

$$\Lambda(z) = \frac{1}{2\alpha}\,[f(z) - X(z)\,\{L(z) - Cz - D\}]. \qquad ...(5.36)$$

6. Case of Normal Pressure

We now consider, as a specific example the case in which the crack is opened by normal pressure $p_1(r) = p_0/r^2$, $p_2(r) = 0$, so that we have

$$f(x) = f_1(x) + i f_2(x) = \frac{\beta^2 - \alpha^2}{4\mu_2}\,p_0 = f_0.$$

Then we have

$$L(z) = f_0[z^2 + 2i\omega(B - A) - \{(A^2 + B^2) + (2(B - A)\omega)^2\}]. \qquad ...(6.1)$$

Hence

$$\Lambda(z) = \frac{f_0}{2\alpha}\left[1 - (z^2 + Cz + D)\,X(z)\right] \qquad ...(6.2)$$

where $X(z)$ is defined in (5.34). From (5.34) we have

(i) for $A < x < B$,

$$\left.\begin{aligned} X^+(x) &= -\, i\, k^{1/2}\{(x^2 - A^2)(B^2 - x^2)\}^{-1/2}(\cos \omega\theta + i \sin \omega\theta) \\ X^-(x) &= i\, k^{-1/2}\{(x^2 - A^2)(B^2 - x^2)\}^{-1/2}(\cos \omega\theta + i \sin \omega\theta) \end{aligned}\right\} \quad \ldots(6.3)$$

where $\theta = \log\{(x - A)(B + x)/(A + x)(B - x)\}$, while for $-B \leqslant x \leqslant -A$, we have

$$\left.\begin{aligned} X^+(x) &= i\, k^{1/2}\{(x^2 - A^2)(B^2 - x^2)\}^{-1/2}(\cos \omega\theta + i \sin \omega\theta) \\ X^-(x) &= -\, i k^{-1/2}\{(x^2 - A^2)(B^2 - x^2)\}^{-1/2}(\cos \omega\theta + i \sin \omega\theta). \end{aligned}\right\} \quad \ldots(6.4)$$

(ii) for $0 \leqslant x < A$,

$$X^+(x) = X^-(x) = -\{(A^2 - x^2)(B^2 - x^2)\}^{-1/2}(\cos \omega\theta_1 + i \sin \omega\theta_1) \quad \ldots(6.5)$$

where $\theta_1 = \log\{(A - x)(B + x)/(A + x)(B - x)\}$, and

(iii) for $x > B$,

$$X^+(x) = X^-(x) = \{(x^2 - A^2)(x^2 - B^2)\}^{-1/2}(\cos \omega\theta_2 + i \sin \omega\theta_2) \quad \ldots(6.6)$$

where $\theta_2 = \log\{(x - A)(x + B)/(x + A)(x - B)\}$.

Hence for $A < x < B$, if we write $C = C_1 + i\, C_2$, $D = D_1 + i\, D_2$ where C_i, D_i, $i = 1, 2$ are arbitrary real constants, then from (5.29) we have

$$\wedge^+(x) - \wedge^-(x) = \frac{i f_0}{\{(\alpha^2 - \beta^2)(x^2 - A^2)(B^2 - x^2)\}^{1/2}} \times [x^2 + (C_1 + iC_2)\, x + D_1 + iD_2](\cos \omega\theta + i \sin \omega\theta) \quad \ldots(6.7)$$

and

$$\wedge^+(x) + \wedge^-(x) = \frac{f_0}{\alpha}\left[1 - \frac{i\beta(x^2 + (C_1 + iC_2) + D_1 + iD_2)}{\{(\alpha^2 - \beta^2)(x^2 - A^2)(B^2 - x^2)\}^{1/2}} \times (\cos \omega\theta + i \sin \omega\theta)\right]. \quad \ldots(6.8)$$

The Pelmelj relation for $A < x < B$, then yields

$$a_1(x) = f_0\left[\frac{(x^2 + C_1 x + D_1)\cos \omega\theta - (C_2 x + D_2)\sin \omega\theta}{\{(\alpha^2 - \beta^2)(x^2 - A^2)(B^2 - x^2)\}^{1/2}}\right] \quad \ldots(6.9)$$

$$b_1(x) = -f_0\left[\frac{(x^2 + C_1 x + D_1)\sin \omega\theta + (C_2 x + D_2)\sin \omega\theta}{\{(\alpha^2 - \beta^2)(x^2 - A^2)(B^2 - x^2)\}^{1/2}}\right] \ldots(6.10)$$

Similarly for $-B < x < -A$ we have

$$a(x) = -f_0\left[\frac{(x^2 + C_1 x + D_1)\cos \omega\theta - (C_2 x + D_2)\sin \omega\theta}{\{(\alpha^2 - \beta^2)(x^2 - A^2)(B^2 - x^2)\}^{1/2}}\right] \quad \ldots(6.9a)$$

$$b(x) = f_0 \left[\frac{(x^2 + C_1 x + D_1) \sin \omega\theta + (C_2 x + D_2) \cos \omega\theta}{\{(\alpha^2 - \beta^2)(x^2 - A^2)(B^2 - x^2)\}^{1/2}} \right]. \quad \ldots(6.10a)$$

The relations $a(x) = -a_1(-x)$ and $b(x) = b_1(-x)$ on $-B < x < -A$ are satisfied provided $C_1 = D_2 = 0$. The other two constants are determined from the continuity of displacement at the end points of the crack. From (3.3), (3.4) and (5.7), (5.8) we have

$$\frac{d}{dr} [(u_\theta(r, 0+) - u_\theta(r, 0-)) + i(u_r(r, 0+) - u_r(r, 0-))]$$
$$= - M^{-1}[\varphi(s) + i\psi(s); r] = - \lambda(r^{n/2}), \text{ for } a < r < b.$$

Hence we have

$$\int_a^b u^{(2/n)-1} \lambda(u)\, du = 0. \quad \ldots(6.11)$$

From (5.11) and (5.12) we have

$$r^2[\sigma_{\theta\theta}(r, 0+) + i\,\sigma_{r\theta}(r, 0+)] = \frac{4\mu_2}{(\alpha^2 - \beta^2)\pi i} \int_L \frac{\lambda(u)}{u - x}\, du \quad \ldots(6.12)$$

where $x = r^{n/2}$. For calculating the stresses the above integral is evaluated with the help of Pelmelj relations for $r < a$ and $r > b$.

Example: To illustrate the method given above we consider the problem of finding the stresses and the stress intensity factors for a pair of coplanar Griffith cracks situated at the interface of two semi-infinite planes having different elastic properties bonded together. The two cracks are defined by the relations $\theta = 0$, $a \leqslant r \leqslant b$, $\theta = \pi$, $a \leqslant r \leqslant b$ and their faces are subjected to prescribed normal pressure p_0/r^2. Because of symmetry in the problem we are required to find the solution of the equations of elasticity for the region $0 \leqslant r < \infty$, $-\pi/2 \leqslant \theta \leqslant \pi/2$. This problem corresponds to $n = 2$ of the general problem considered above. Since $n = 2$, we have $A = a$, $B = b$, and $x = r$. With the help of the integrals given in section 4 and eqn. (6.11) we have

$$I_2 + iC_2 I_1 + D_1 I_0 = 0. \quad \ldots(6.13)$$

By equating the real and imaginary parts we get

$$C_2 = - I_2''/I_1', \; D_1 = - (I_1'' I_2'' + I_1' I_2')/I_0 I_1'.$$

For $r > b$, we have

$$r^2 [\sigma_{\theta\theta}(r, 0+) + i\sigma_{r\theta}(r, 0+)]$$
$$= - p_0 \left[1 - \frac{i\beta(r^2 + irC_2 + D)(\cos \omega\theta_2 + i \sin \omega\theta_2)}{\{(\alpha^2 - \beta^2)(r^2 - a^2)(r^2 - b^2)\}^{1/2}} \right]. \quad \ldots(6.14)$$

For $r < a$, we have

$$r^2[\sigma_{\theta\theta}(r, 0+) + i\,\sigma_{r\theta}(r, 0-)]$$

$$= -p_0\left[1 + \frac{i\beta(r^2 + irC_2 + D_1)(\cos\omega\theta_1 + i\sin\omega\theta_1)}{\{(\alpha^1 - \beta^2)(a^2 - r^2)(b^2 - r^2)\}^{1/2}}\right]. \quad \ldots(6.15)$$

The stress intensity factor at the crack tip $\theta = 0$, $r = b$ is given by the equation

$$N'_b + iN''_b = \lim_{r\to b}[(r - b)^{1/2}\{\sigma_{\theta\theta}(r, 0+) + i\sigma_{r\theta}(r, 0-)\}\exp(-i\omega\theta_2)]_{r>b}$$

$$= \frac{i\beta p_0}{b^2(\alpha^2 - \beta^2)\,2b(b^2 - a^2)\}^{1/2}}(b^2 + i\,b\,C_2 + D_1). \quad \ldots(6.16)$$

Similarly at the tip $\theta = 0$, $r = a$ we have

$$N'_a + iN''_a = -\frac{i\beta p_0(a^2 + iaC_2 + D_1)}{a^2\{(\alpha^2 - \beta^2)\,2a(b^2 - a^2)\}^{1/2}}. \quad \ldots(6.17)$$

The case when $n = 1$ can be treated in a similar way.

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