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REMARK ON SOME THEOREMS OF K. ISEKI

by HAIMABATI CHATTERJEE, 100, *Kalibazar, P.O. & Dist. Burdwan, West Bengal*

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In this paper, we shall prove fixed point theorems for continuous densifying mapping. Our results extend some recent results of Iseki (1976).

Let A be a bounded subset of a complete metric space X. By the real number $\alpha(A)$, we denote the infimum of all positive numbers ϵ such that A admits a finite covering consisting of subsets with diameter less than ϵ (*see* Kuratowski 1952).

Then one has the following (Iseki 1976)

(1) $0 \leqslant \alpha(A) \leqslant D(A)$ where $D(A)$ is the diameter of A.

(2) $\alpha(A) = 0 \Leftrightarrow A$ is precompact.

(3) $\alpha(A \cup B) = \max \{\alpha(A), \alpha(B)\}$.

Furi and Vignoli (1965, 1969) introduced the notion of densifying mapping defined as follows:

A mapping T is called densifying, if for any bounded subset A of X with $\alpha(A) > 0$, we have $\alpha(TA) < \alpha(A)$.

Recently Iseki (1976) obtained the following theorems.

Theorem KI1 — Let T be a continuous, densifying mapping of a bounded complete metric space X into itself. Let F be a real-valued continuous function defined on $X \times X$. If for every $x, y \in X$ with $x \neq Tx$, $y \neq Ty$

$$F(Tx, Ty) < \tfrac{1}{2} [F(x, Tx) + F(y, Ty)] \qquad \text{...(1)}$$

then T has a fixed point.

Theorem KI2 — Let T be a continuous, densifying mapping of a bounded complete metric space (X, d) into itself. If for every $x, y \in X$ with $x \neq Tx$, $y \neq Ty$

$$d(Tx, Ty) < \tfrac{1}{2} [d(x, Tx) + \tfrac{1}{2} d(y, Ty)]$$

then T has a fixed point.

Theorem KI3 — Let T be a continuous densifying mapping of a bounded complete metric space (X, d) into itself. If for every distinct $x, y \in X$

$$d(Tx, Ty) < \tfrac{1}{2}[d(x, Ty) + d(y, Tx)] \qquad \ldots(2)$$

then T has a unique fixed point.

The object of this note is to extend the above theorems of Iseki. We have the following theorems:

Theorem 1 — Let T be a continuous, densifying mapping of a bounded complete metric space X into itself and let F be a real valued continuous function defined on $X \times X$. If for every x, y in X with $x \neq Tx$, $y \neq Ty$, $x \neq y$

$$F(Tx, Ty) < \max\{F(x, y), \tfrac{1}{2}[F(x, Tx) + F(y, Ty)]\} \qquad \ldots(3)$$

then T has a fixed point.

We use Furi-Vignoli argument to prove Theorem 1.

PROOF: For arbitrary $x_0 \in X$, consider the sequence of iterates $\{x_n\}$ where $x_{n+1} = Tx_n$, $n = 0, 1, 2, \ldots$ and let $A = \bigcup_{n=0}^{\infty} x_n$. Then $TA = \bigcup_{n=1}^{\infty} x_n \subset A$ and by the continuity of T, we have $T(\bar{A}) \subset \overline{T(A)} \subset \bar{A}$.

Hence $\bar{A}$ is invariant under T and is bounded. To prove that A is compact, suppose $\alpha(A) > 0$. Since T is densifying,

$$\alpha(T(A)) < \alpha(A).$$

On the other hand

$$A = \{x_0\} \cup T(A)$$

and so

$$\alpha(A) = \max\{\alpha\{x_0\}, \alpha(T(A))\}$$
$$= \max\{0, \alpha(T(A))\}.$$

Hence we have $\alpha(A) = 0$

i.e. A is precompact. Now $\bar{A}$ is a complete metric space and A is precompact, so $\bar{A}$ is compact. Since the function $\psi(x) = F(x, Tx)$ is continuous on $\bar{A}$, $F(x, Tx)$ has minimum points in $\bar{A}$. Let ξ be the minimum point i.e. $\psi(\xi) = F(\xi, T\xi)$ is minimum. Then ξ or $T\xi$ is a fixed point of T. Suppose $\xi \neq T\xi$ and $T\xi \neq T^2\xi$. Then by (2)

$$\psi(T\xi) = F(T\xi, T^2\xi)$$
$$< \max\{F(\xi, T\xi), \tfrac{1}{2}[F(\xi, T\xi) + F(T\xi, T^2\xi)]\}$$

and this implies $\psi(T\xi) < \psi(\xi)$, which contradicts the minimality of $\psi(\xi)$. Therefore ξ or $T\xi$ is a fixed point of T. From the above proof, we obtain the following which extends Theorem KI2.

Theorem 2 — Let T be a continuous, densifying mapping of a bounded complete metric space (X, d) into itself. If for every x, y in X $x \neq Tx$, $y \neq Ty$, $x \neq y$,

$$d(Tx, Ty) < \max \{d(x, y), \tfrac{1}{2} [d(x, Tx) + d(y, Ty)]\}$$

then T has a fixed point.

The following theorem extends Theorem KI3.

Theorem 3 — Let T be a continuous, densifying mapping of a bounded complete metric space (X, d) into itself. If for every distinct x, y in X

$$d(Tx, Ty) < \max \{d(x, y), \tfrac{1}{2} [d(x, Tx) + d(y, Ty)], \tfrac{1}{2} [d(x, Ty) + d(y, Tx)]\} \quad \ldots(4)$$

then T has a unique fixed point.

PROOF : For arbitrary $x_0 \in X$, let $x_{n+1} = Tx_n$, $n = 0, 1, 2, \ldots$ and $A = \bigcup_{n=0}^{\infty} x_n$. Then A is bounded, $TA \subset A$ and as in Theorem 1, $\bar{A}$ is invariant under T and is bounded. Suppose $\alpha(A) > 0$ then

$$\alpha(TA) < \alpha(A), \; A = \{x_0\} \cup T(A) \text{ implies}$$

$$\alpha(A) = \max \{\alpha(T(A)), \alpha(x_0)\} = \alpha(T(A).$$

Hence $\alpha(A) = 0$, and $\bar{A}$ is compact. Since $d(x, Tx)$ is continuous on $\bar{A}$, $d(x, Tx)$ has a minimum point ξ in $\bar{A}$. Then ξ is a fixed point of T. Suppose $\xi \neq T\xi$. Then by (4), we have

$$d(T\xi, T^2\xi) < \max \{d(\xi, T\xi), \tfrac{1}{2} [d(\xi, T\xi) + d(T\xi, T^2\xi)], \tfrac{1}{2} [d(\xi, T^2\xi)]\}$$

and this implies

$$d(T\xi, T^2\xi) < d(\xi, T\xi)$$

i.e. ξ is not a minimum point. Hence ξ is a fixed point of T. If $\xi, \eta, \xi \neq \eta$, be two fixed points of T, then by (4) we have

$$d(\xi, \eta) = d(T\xi, T\eta) < \max \{d(\xi, \eta), 0, d(\xi, \eta)\}$$

which implies $\xi = \eta$. Hence ξ is a unique fixed point of T. This completes the proof.

Remarks: The assumptions 'bounded' of Theorems 1, 2, 3 may be replaced by "for some $x_0 \in X$, the sequence of iterates of x_0 is bounded" and in Theorem 1, the continuity of F may be replaced by the lower semi-continuity of F.

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