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BOUNDED INDEX AND SUMMABILITY METHODS

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In this paper the author has discussed matrix transformations between sequences of bounded index.

INTRODUCTION

An entire function f is said to be of bounded index, if there exists a non-negative integer N such that

$$\max_{0 \leqslant j \leqslant N} \left\{ \frac{| f^{(j)}(z) |}{j!} \right\} \geqslant \frac{| f^{(k)}(z) |}{k!}$$

for all z and for all k. The least such integer N is called the index of f.

A sequence $x = \{x_k\}$ of complex numbers is an entire sequence, if $\sum\limits_{k=0}^{\infty} | x_k | q^k$ converges for every positive integer q.

An entire sequence $x = \{x_k\}$ is of bounded index, if $f(z) = \sum\limits_{k=0}^{\infty} x_k z^k$ is an entire function of bounded index (*see* Fricke and Powel 1976).

The following notations are employed.

C_0 = the space of all complex null sequences. It is a BK-space, with norm

$$\| x \| = \operatorname*{Sup}_{(k)} | x_k | , \forall x = \{x_k\} \in C_0.$$

C = the space of all complex convergent sequences.

Γ = the space of all sequences $x = \{x_k\}$, with $| x_k |^{1/k} \to 0$, as $k \to \infty$.

Γ^* = the space of sequences $x = \{x_k\}$ such that $\{ | x_k |^{1/k}\}$ is bounded. The space Γ^* may be regarded as, the space conjugate to Γ, regarded as the space of integral functions $f(z) = \sum\limits_{k=1}^{\infty} x_k z^k$.

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χ = the set of all sequences $x = \{x_k\}$, such that, $(k! \mid x_k \mid)^{1/k} \to 0$, as $k \to \infty$. χ can be regarded as the collection of all entire functions $f(z) = \sum_{k=1}^{\infty} x_k z^k$, of exponential type of order '1' and type '0' (*see* Kamthan 1976).

A matrix $A = (a_{nk})$ is a $C_0 - \Gamma$ method, if it transforms C_0 into Γ; similarly, $C - \Gamma$ method, $C_0 - \chi$ method, $C - \Gamma^*$ method, and $l - \Gamma^*$ method are defined.

For an entire function $f(z)$ and a sequence $\{z_i\}$ of complex numbers, define the matrix transformation,

$$A(f, z_i) = (a_{nk}) \text{ by } f(z) = \sum_{k=1}^{\infty} a_{nk}(z - z_n)^k \text{ for } n = 1, 2, \ldots .$$

Then, we have $a_{nk} = \dfrac{f^{(k)}(z_n)}{k!}$, $(n, k = 1, 2, \ldots)$ and the matrix $A'(f, z_i) = (b_{nk})$ which is defined by

$$f(z) = \sum_{n=1}^{\infty} b_{nk}(z - z_k)^n, \text{ for } k = 1, 2, \ldots .$$

The matrix $A'(f, z_i) = (b_{nk})$ is the transpose of $A(f, z_i) = (a_{nk})$. That is, $a_{n,k} = b_{k,n}$, $(n, k = 1, 2, \ldots)$ (*see* Fricke and Powel 1976).

Some Known Theorems

Theorem 1 — Let $y_n = \sum_{k=1}^{\infty} a_{nk}x_k$, $(n = 1, 2, \ldots)$. In order that $\{y_n\}$ should belong to Γ, whenever $\{x_k\}$ belongs to C_0, it is necessary and sufficient that the sequence $\{\theta_n\}$ is a null sequence, where $\theta_n = (\sum_{k=1}^{\infty} \mid a_{nk} \mid)^{1/n}$, $(n = 1, 2, \ldots)$ (*see* Rao 1969).

Theorem 1 holds even if C_0 is replaced by C (*see* Rao 1969).

Theorem 2 — Let $y_n = \sum_{k=1}^{\infty} a_{nk}x_k$. In order that $\{y_n\}$ should belong to Γ^*, whenever $\{x_k\}$ belongs to C, it is necessary and sufficient that the sequence $\{\theta_n\}$ is bounded, where θ_n $(n = 1, 2, \ldots)$ are given as in Theorem 1 (*see* Rao 1969).

Theorem 3 — Let $y_n = \sum_{k=1}^{\infty} a_{nk}x_k$. In order that $\{y_n\}$ should belong to χ, whenever $\{x_k\}$ belongs to C_0, it is necessary and sufficient that $\sigma_n \to 0$, as $n \to \infty$, where $\sigma_n = (n! \sum_{k=1}^{\infty} \mid a_{nk} \mid)^{1/n}$ $(n = 1, 2, \ldots)$ (*see* Sridhar 1978).

Results and Proofs

We prove Theorem I only as follows:

Theorem I — If $f(z)$ is an entire function of bounded index, then for any sequence $\{z_k\}_1^\infty$, $A'(f, z_k) = (b_{nk})$ is a $C_0 - \Gamma$ method, if and only if,

$$\left(\frac{|f^{(n)}(z_k)|}{n!}\right)^{1/n} \to 0, \text{ as } n \to \infty, (k = 1, 2, \ldots). \quad \ldots(1)$$

PROOF : Let $A = (a_{nk})$ be a $C_0 - \Gamma$ method. By Theorem 1 we have

$$\left(\sum_{k=1}^{\infty} |a_{nk}|\right)^{1/n} \to 0, \text{ as } n \to \infty, (k = 1, 2, \ldots) \quad \ldots(2)$$

Let $A'(f, z_k) = (b_{nk})$ be a $C_0 - \Gamma$ method.

Then given $\epsilon > 0$, by (2), there exists n_0 such that

$$\left(\sum_{k=1}^{\infty} |b_{nk}|\right)^{1/n} < \epsilon, \forall\, n \geqslant n_0, \forall\, k = 1, 2, \ldots. \quad \ldots(3)$$

But we have $|b_{nk}| = \dfrac{|f^{(n)}(z_k)|}{n!}$.

That is, we have that $\left(\dfrac{|f^{(n)}(z_k)|}{n!}\right)^{1/n} \to 0$, as $n \to \infty$, $\forall\, k = 1, 2, \ldots$.

Now, let f be an entire function of bounded index and let $\{z_k\}_1^\infty$ be a sequence such that (1) holds. Since $f(z)$ is of bounded index, we have that $f(3z)$ is of bounded index (*see* Fricke 1972). Let N be the index of $f(3z)$.

Hence, $\max\limits_{0 \leqslant j \leqslant N} \left\{\dfrac{|f^{(j)}(z)|}{j!}\right\} \geqslant 3^{n-N} \dfrac{|f^{(n)}(z)|}{n!}$, $\forall\, z$ and $\forall\, k$. Therefore

$$\left(\sum_{k=1}^{\infty} |b_{nk}|\right)^{1/n} = \left(\sum_{k=1}^{\infty} \frac{|f^{(n)}(z)|}{n!}\right)^{1/n}$$

$$\leqslant \left[\sum_{k=1}^{\infty} 3^{N-n} \max_{0 \leqslant j \leqslant N} \left\{\frac{|f^{(j)}(z_k)|}{j!}\right\}\right]^{1/n}$$

$$\to 0, \text{ by (1)}.$$

Hence $A'(f, z_k) = (b_{nk})$ is a $C_0 - \Gamma$ method. ∎

Similarly the following results are true.

Theorem II — For $C - \Gamma$ method also, Theorem I holds.

Theorem III — If $f(z)$ is a entire function of bounded index, then for all sequences $\{z_k\}$, $A'(f, z_k) = (b_{nk})$ is an $C_0 - \chi$ method, if and only if,

$$(\mid f^{(n)}(z_k) \mid)^{1/n} \to 0, \text{ as } n \to \infty, \forall\, k = 1, 2, \dots .$$

Theorem IV — If $f(z)$ is an entire function of bounded index, then for all sequences $\{z_k\}$, $A'(f, z_k) = (b_{nk})$ is a $C - \Gamma^*$ method, if and only if, $\left\{\frac{\mid f^{(n)}(z_k) \mid}{n!}\right\}^{1/n}$ is bounded for all n.

Since $A'(f, z_k) = (b_{nk})$ is the transpose of the matrix $A(f, z_k) = (a_{nk})$, the following theorem is also true.

Theorem V — If $f(z)$ is a entire function of bounded index, then for any sequence $\{z_n\}_1^{\infty}$, $A(f, z_n) = (a_{nk})$ is a $C_0 - \Gamma$ method, if and only if, $\left(\frac{\mid f^{(k)}(z_n) \mid}{k!}\right)^{1/n} \to 0$, as $n \to \infty$, $\forall\, k = 1, 2, \dots$.

The Sonnenschein matrix $A(f) = (a_{nk})$ has been studied by many people including Meyer-König (1950). It is defined by, $[f(z)]^n = \sum_{k=0}^{\infty} a_{nk} z^k$, (for $n \geqslant 1$) where f is analytic at $z = 0$ and $a_{00} = 1$, $a_{0k} = 0$, for $k \geqslant 1$.

Theorem VI — The Sonnenschein matrix $A(f)$ is an $l - \Gamma^*$ method, if and only if, $f(0) = 0$.

PROOF : Suppose that $f(0) \neq 0$. Choose an integer $q > 0$ such that $\mid f(0) \mid q > 1$. Then, for $M = M(q)$, we have

$$\mid a_{nk} \mid^{1/n} \leqslant M, \forall\, n = 1, 2, \dots . \quad (\textit{see} \text{ Rao 1970}) \qquad \dots(4)$$

However, there exists an integer N such that, if $n \geqslant N$,

$$\mid f(0) \mid^n q^n > M'^n$$

i.e.

$$\mid a_{n0} \mid^{1/n} > M \text{ where } M = \left(\frac{M'}{q}\right).$$

This is a contradiction to (4).

Therefore, it follows that $f(0) = 0$.

'$\Leftarrow$' If $f(0) = 0$, then $A(f) = (a_{nk})$ is upper triangular (*see* Fricke and Powel 1976). Since, $f(z)$ is analytic at $z = 0$; there exists $R > 0$, such that $f(z)$ is analytic on $\{z : \mid z \mid < 2R\}$ (hence $[f(z)]^n$ is analytic on $\{z: \mid z \mid < 2R\}$). Let $\Gamma = \{t : \mid t \mid = R\}$. There exists $M > 0$, such that, Sup $\{\mid f(z) \mid : t \in \Gamma\} \leqslant M$.

So by Cauchy's integral formula, we have

$$| a_{nk} | = \left| \frac{1}{2\pi i} \int_{\Gamma} \frac{[f(z)]^n}{t^{k+1}} dt \right| \leqslant R^{-k} M^n, \ \forall \ k \geqslant n > 0. \qquad \ldots(5)$$

Take $R = 2M$ in (5). Then we have

$$| a_{nk} |^{1/n} \leqslant M \quad (n = 1, 2, \ldots).$$

Hence $A(f)$ is an $l - \Gamma^*$ method by Theorem 3. ∎

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