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THE GENERALIZED CAUCHY EQUATION FOR SELF-ADJOINT OPERATORS ON A HILBERT SPACE

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In this paper the functional equation $S(s + t) = H(S(s), S(t))$, $s, t \geqslant 0$, for self-adjoint operators on a Hilbert space is discussed in the uniform operator topology, and an existence and uniqueness theorem for the solution of the equation is obtained under the conditions of symmetry, monotonicity and associativity.

1. INTRODUCTION

A generalization of the Cauchy functional equation $f(x + y) = f(x)f(y)$ may be put in the form $f(x + y) = H(f(x), f(y))$, where f is a real or complex valued function of a real or complex variable and H is a real or complex valued function of two real or complex variables (Aczél 1966a, b). Under the conditions of internness and strict monotonicity, the uniqueness of the solution of the generalized equation in the scalar case was investigated by Aczél (1964, 1966b) and Aczél and Hosszú (1965). In the operator-valued case, the analogue of the Cauchy functional equation is the semi-group equation $T(s + t) = T(s)\, T(t)$, $T(0) = I$ (the identity operator), $s, t \geqslant 0$ and $\{T(t)\}$ is a family of bounded linear operators on a Banach space. The extensive treatment of operator semi-groups is given by Hille and Phillips (1957), who also mention some generalizations of the semi-group equation and their holomorphic solutions. In this paper we have studied the existence and uniqueness of the solution of the generalized Cauchy equation for positive self-adjoint operators on a Hilbert space under the conditions of strict monotonicity as in Aczél (1966b), and with two boundary conditions as in Aczél and Hosszú (1965). We remark that the regularity properties of the solution of the generalized Cauchy equation under Lipschitzian type conditions were studied by Buche and Vasudeva (1976).

2. THE MAIN THEOREM

Let $\mathscr{H}$ denote a Hilbert space, and $\mathscr{P}(\mathscr{H})$ the class of bounded positive self-adjoint linear operators on $\mathscr{H}$. Let $R^+ = [0, \infty)$. Let $\{S(t); t \in R^+\}$, $S : R^+ \to \mathscr{P}(\mathscr{H})$, $S(0) = I$, be a one-parameter family of operators on $\mathscr{H}$. The family $\{S(t)\}$ is a 'generalized Cauchy system' of operators, if it satisfies the equation

$$S(s + t) = H(S(s), S(t)),\ s, t \in R^+ \qquad ...(1)$$

where $H : \mathscr{P}(\mathscr{H}) \times \mathscr{P}(\mathscr{H}) \to \mathscr{P}(\mathscr{H})$ is a function.

In what follows we use the concept of monotonicity of functions of one and two real variables (cf. Hobson 1927). For the properties of self-adjoint operators on a Hilbert space, reference may be made to Riesz and Sz.-Nagy (1952).

Theorem — Let $E \in \mathcal{P}(\mathcal{H})$, $E > I$, and let $\{E_\lambda, m \leqslant \lambda \leqslant M, m \geqslant 1\}$ be the decomposition of the identity operator corresponding to the operator E. Let $h : R^+ X R^+ \to R^+$ be a function of two variables, such that

(i) h is continuous on $R^+ \times R^+$,

(ii) $h(x, y) = h(y, x)$, $x, y \in R^+$ (symmetry),

(iii) $h(1, y) = y, y \in R^+$,

(iv) h is strictly monotone increasing in one as well as two variables.

Let $H : \mathcal{P}(\mathcal{H}) \times \mathcal{P}(\mathcal{H}) \to \mathcal{P}(\mathcal{H})$ be a function defined by

$$H(A, B) = h(f(E), \varphi(E)) \qquad \ldots(2)$$

where $A = f(E)$, $B = \varphi(E)$ are continuous functions of $E, f(E) \in \mathcal{P}(\mathcal{H}), \varphi(E) \in \mathcal{P}(\mathcal{H})$. Then there exists a strictly monotone increasing family of operators $\{S(t); t \in R^+\}$, $S : R^+ \to \mathcal{P}(\mathcal{H})$, satisfying the conditions $S(0) = I$, $S(1) = E$.

If, moreover, the associativity hypothesis

$$h(x, h(y, z)) = h(y, h(x, z)), \; x, y, z \in R^+ \qquad \ldots(3)$$

holds, then $\{S(t)\}$ satisfies eqn. (1).

Let $g : R^+ \to R^+$, $g(x) = h(x, x)$, $x \in R^+$, and $g^{(-n)}$ be the n-fold inverse $g^{-1}(g^{-1}(\ldots(g^{-1})))$ (n times), $n = 1, 2, \ldots$. If $g^{(-n)}(a) \to 1$ as $n \to \infty$, for every real $a > 0$, then the solution $\{S(t)\}$ is continuous in the uniform operator topology, and the continuous strictly increasing solution of eqn. (1) is unique.

PROOF: The proof is based on the properties of continuous strictly monotone functions of one and two real variables and on the spectral representation of continuous functions of positive self-adjoint operators on a Hilbert space.

Given

$$S(1) = E = \int_m^M \lambda \, dE_\lambda$$

we define

$$S(1/2) = \int_m^M g^{-1}(\lambda) \, dE_\lambda$$

so as to have

$$H(S(1/2), S(1/2)) = \int_m^M h(g^{-1}(\lambda), g^{-1}(\lambda))\, dE_\lambda$$

$$= \int_m^M \lambda\, dE_\lambda = S(1)$$

since $\lambda = g(g^{-1}(\lambda)) = h(g^{-1}(\lambda), g^{-1}(\lambda)), m \leqslant \lambda \leqslant M$.

Then $\lambda \geqslant m \geqslant 1 = h(1, 1) = g(1)$ implies $g^{-1}(\lambda) \geqslant 1$, which implies $S(1/2) \geqslant I$. Also $\lambda \geqslant 1 \Rightarrow h(\lambda, \lambda) \geqslant h(1, \lambda)$, since h is monotone in one variable, when the other is fixed. Hence $g(\lambda) \geqslant \lambda$, or $\lambda \geqslant g^{-1}(\lambda)$, and, therefore, $S(1) \geqslant S(1/2)$.

Inductively, we define

$$S(1/2^n) = \int_m^M g^{(-n)}(\lambda)\, dE_\lambda,\ n = 2, 3, \ldots . \qquad \ldots(4)$$

This determines the family $\{S(t)\}$ for all points t of the type $1/2^n$, $n = 0, 1, 2, \ldots$, in such a way that

$$S(0) = I \leqslant S(1/2^{n+1}) \leqslant S(1/2^n) \leqslant E = S(1).$$

We observe that

$$H(S(1/2^{n+1}), S(1/2^n)) = \int_m^M h(g^{-(n+1)}(\lambda), g^{-(n)}(\lambda))\ dE_\lambda.$$

Now if $m \leqslant x \leqslant y \leqslant M$, we can, by using the Intermediate Value Theorem, find z such that $x < z < y$ and $h(z, z) = h(x, y)$. Thus one can find a continuous function $u(\lambda)$, $m \leqslant \lambda \leqslant M$, such that

$$H(S(1/2^{n+1}), S(1/2^n)) = \int_m^M h(u(\lambda), u(\lambda))\, dE_\lambda.$$

We then define the value of S at the mid-point of the interval $(1/2^{n+1}, 1/2^n)$ by $\int_m^M u(\lambda)\ dE_\lambda$.

This process of determining $S(t)$ at mid-points of the subintervals of $[0, 1]$, at the end-points of which $S(t)$ is already determined, is carried out indefinitely till $S(t)$ is determined at all points $t_n \in [0, 1]$, where t_n is of the type

$$(\alpha_1/2) + (\alpha_2/2^2) + \ldots + (\alpha_n/2^n),\ \alpha_i = 0, 1,\ i = 1, 2, \ldots, n.$$

At other points $t \in [0, 1]$, $\{S(t)\}$ is determined by taking a limit from the left along points of the type t_n. If we select two such different sequences $\{t_n^{(1)}\}$, $\{t_n^{(2)}\}$ in $[0, 1]$, each of which converges to t, then for every member v_n of one of the sequences, we can find a member w_m of the other sequence such that $n \leqslant m$, and $v_n \leqslant w_m$, and hence $S(v_n) \leqslant S(w_m)$. Thus the value of the limit will not depend upon the choice of the monotonic increasing sequence $\{t_n\}$ converging to any given $t \in [0, 1]$. The construction so far has given us $\{S(t)\}$ as a strictly increasing continuous from the left operator-valued function for $0 \leqslant t \leqslant 1$.

For $t \geqslant 1$, we define

$$S(n) = H(S(1), S(n-1))$$

and

$$S(n+\tau) = H(S(n), S(\tau))$$

$n = 2, 3, \ldots, 0 \leqslant \tau \leqslant 1$. This determines $\{S(t)\}$, for $t \geqslant 0$, as a strictly monotone increasing continuous from the left family in $\mathcal{P}(\mathcal{H})$, satisfying the condition $S(0) = I$, $S(1) = E$.

Now the associativity condition, along with symmetry, implies

$$h(h(x, y), h(z, w)) = h(h(x, z), h(y, w))$$

$x, y, z, w \in R^+$, which are also equal to similar expressions obtained by permuting the letters x, y, z, w.

This leads to

$$\begin{aligned} S(m+n+\tau+\sigma) &= H(S(m+n), S(\tau+\sigma)) \\ &= H(S(m+\tau), S(n+\sigma)) \end{aligned}$$

for all $m, n = 0, 1, 2, \ldots, 0 \leqslant \tau \leqslant 1, 0 \leqslant \sigma \leqslant 1$; here in case $\tau + \sigma > 1$, we will have to further write $S(\tau+\sigma) = H(S(1), S(\tau+\sigma-1))$ and make one more application of the associativity hypothesis and the condition (iii) on h. Thus, under the additional associativity hypothesis, $\{S(t)\}$ will satisfy eqn. (1).

Lastly, we examine the effect of the condition $g^{(-n)}(a) \to 1$ as $n \to \infty$, for every real $a > 0$. This condition implies that $S(1/2^n) \to I$ as $n \to \infty$. Since for $\tau > 0$, $t \geqslant 0$, $S(t+\tau) - S(t) = H(S(t), S(\tau)) - H(S(t), S(0))$, the norm of the difference will tend to 0 as $\tau \to 0$, in view of the continuity of H and the fact that

$$S(0) \leqslant S(\tau) \leqslant S(1/2^n), \text{ for } 0 \leqslant \tau \leqslant 1/2^n.$$

Thus the one-parameter family of operators $\{S(t)\}$ will be continuous from the right at all $t \geqslant 0$. This, along with continuity from the left, yields continuity in the uniform operator topology.

In the above discussion the inverse function g^{-1} and its iterates exist uniquely because of the conditions of strict monotonicity and continuity imposed on the function h. The conditions imposed on H and all other hypotheses taken together now give us the uniqueness of the solution of eqn. (1).

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