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# ON THE LOGARITHMIC ORDER OF ANALYTIC FUNCTIONS REPRESENTED BY DIRICHLET SERIES

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In the present paper, to study precisely the growth of functions of zero order which are represented by Dirichlet series and are analytic in the half plane, the concepts of logarithmic order and lower logarithmic order have been introduced and the growth of such functions has been studied through these concepts.

§1. Consider the Dirichlet series

$$f(s) = \sum_{n=1}^{\infty} a_n \exp(s\lambda_n) \quad \ldots(1.1)$$

where $\lambda_1 \geqslant 0$, $0 < \lambda_n < \lambda_{n+1} \to \infty$, $s = \sigma + it$ ($\sigma$, $t$ being real variables), $\{a_n\}_1^{\infty}$, is sequence of complex numbers and

$$\limsup_{n \to \infty} \frac{n}{\lambda_n} = D < \infty. \quad \ldots(1.2)$$

If the series given by (1.1) converges absolutely in the half plane Re $s < \alpha$ ($-\infty < \alpha < \infty$) then it is known (Mandelbrojt 1944, p. 166) that the series (1.1) represents an analytic function in Re $s < \alpha$, and since (1.2) holds we have

$$\alpha = -\limsup_{n \to \infty} \frac{\log |a_n|}{\lambda_n}. \quad \ldots(1.3)$$

Set,
$$M(\sigma, f) \equiv M(\sigma) = \underset{-\infty < t < \infty}{\text{l.u.b.}} |f(\sigma + it)|$$

$$m(\sigma, f) \equiv m(\sigma) = \max_{n \geqslant 1} \{ |a_n| e^{\sigma\lambda_n} \}$$

and
$$N(\sigma) = \max \{ n : m(\sigma) = |a_n| e^{\sigma\lambda_n} \}.$$

It is known (Doetsch 1920) that log $M(\sigma)$ is an increasing convex function of $\sigma < \alpha$. Further, if $\alpha = \infty$, $f(s)$ the sum function of (1.1) represents an entire function and in this case much more is known about the growth of $M(\sigma)$, $m(\sigma)$ and $N(\sigma)$. However if $f(s)$ is not an entire function but represents an analytic function in the half plane $\text{Re}\, s < \alpha(-\infty < \alpha < \infty)$, Krishna Nandan (1973) has defined the order $\rho$ and lower order $\lambda(0 \leqslant \lambda, \rho \leqslant \infty)$ of $f(s)$ as

$$\lim_{\sigma \to \alpha} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log \log M(\sigma)}{\log (1 - e^{\sigma - \alpha})^{-1}} = \begin{matrix} \rho \\ \lambda \end{matrix}.$$

Further, if $f(s)$ is of order $\rho(0 < \rho < \infty)$ the type $T$ and lower type $t$ $(0 \leqslant t, T \leqslant \infty)$ of $f(s)$ are defined as,

$$\lim_{\sigma \to \alpha} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log M(\sigma)}{(1 - e^{\sigma - \alpha})^{-\rho}} = \begin{matrix} T \\ t \end{matrix}.$$

The above definition of type and lower type as given by Krishna Nandan does not seem to be feasible if $f(s)$ is of order zero and, therefore, to study precisely the growth of $f(s)$, we introduce the concepts of logarithmic order and lower logarithmic order and derive some of the growth properties of $f(s)$ in the present paper.

§2. Let $D^*_\alpha$ denote the class of all functions $f(s)$ of the form (1.1), which satisfy (1.2) and are analytic in the half plane $\text{Re}\, s < \alpha$ $(-\infty < \alpha < \infty)$ and are of order zero.

Let $A^*_\alpha \subset D^*_\alpha$. We shall say that $f(s) \in A^*_\alpha$, if and only if, there exists $\epsilon > 0$ such that

$$\frac{m(\sigma, f)}{(1 - e^{\sigma - \alpha})^{-\epsilon}} \to \infty, \text{ as } \sigma \to \alpha. \qquad \ldots(2.1)$$

We now define the logarithmic order and lower-logarithmic order of the function $f(s) \in A^*_\alpha$ as follows :

*Definition* — The function $f(s) = \sum_{n=1}^{\infty} a_n \exp(s\lambda_n)$ of the class $A^*_\alpha \subset D^*_\alpha$ is said to be of logarithmic order $\rho^*$ and lower logarithmic order $\lambda^*$ $(1 \leqslant \lambda^*, \rho^* \leqslant \infty)$ in the half-plane $\text{Re}\, s < \alpha$, if

$$\lim_{\sigma \to \alpha} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log \log M(\sigma)}{\log \log (1 - e^{\sigma - \alpha})^{-1}} = \begin{matrix} \rho^* \\ \lambda^* \end{matrix}. \qquad \ldots(2.2)$$

*Remark* 1 : Since $M(\sigma) \geqslant m(\sigma)$, it is easily seen in view of (2.1) that $1 \leqslant \lambda^*$, $\rho^* \leqslant \infty$.

*Remark* 2 : Condition (2.1) is necessary in order that $1 \leqslant \lambda^*, \rho^* \leqslant \infty$. For example, consider the function

$$f(s) = \sum_{n=1}^{\infty} \frac{e^{ns}}{n}.$$

It is seen that

$$M(\sigma, f) = \log (1 - e^{\sigma})^{-1}$$

and that $f(s)$ represents an analytic function in the half-plane Re $s < 0$.

But

$$\frac{\log \log M(\sigma, f)}{\log \log (1 - e^{\sigma})^{-1}} = \frac{\log \log \log (1 - e^{\sigma})^{-1}}{\log \log (1 - e^{\sigma})^{-1}} \to 0, \text{ as } \sigma \to 0$$

so that $\rho^* = \lambda^* = 0$. This is so, because condition (2.1) does not hold.

In the present paper, first we prove a theorem which connects $\rho^*$, $\lambda^*$ with the maximum term $m(\sigma)$ of the series (1.1) and then we obtain formula for $\rho^*$ in terms of the coefficients $a_n$'s and exponents $\lambda_n$'s. Next we construct an example of a function for which $\rho^* > \lambda^*$. Finally in section 4 we obtain formula for $\rho^*$ in terms of $\lambda_{N(\sigma)}$, where $N(\sigma)$ is the rank of the maximum term.

§3. *Theorem* 3.1 — If $f(s) = \sum_{n=1}^{\infty} a_n \exp (s\lambda_n)$ belongs to the class $A^*_\alpha$ and is of logarithmic order $\rho^*(1 < \rho^* < \infty)$, lower logarithmic order $\lambda^*(1 \leqslant \lambda^* < \infty)$ and satisfies

$$\liminf_{n \to \infty} (\lambda_{n+1} - \lambda_n) = \beta > 0 \qquad \ldots(3.1)$$

then

$$\lim_{\sigma \to \alpha} \frac{\sup}{\inf} \frac{\log \log m(\sigma)}{\log \log (1 - e^{\sigma - \alpha})^{-1}} = \frac{\rho^*}{\lambda^*}. \qquad \ldots(3.2)$$

PROOF : From (2.2), for any $\epsilon > 0$ and $\alpha > \sigma > \sigma_1(\epsilon)$ we have

$$\log M(\sigma) < [\log (1 - e^{\sigma - \alpha})^{-1}]^{\rho^* + \epsilon}. \qquad \ldots(3.3)$$

Now $m(\sigma) \leqslant M(\sigma)$, and so

$$\log m(\sigma) \leqslant \log M(\sigma) < [\log (1 - e^{\sigma - \alpha})^{-1}]^{\rho^* + \epsilon}, \text{ for } \sigma > \sigma_1(\epsilon). \qquad \ldots(3.4)$$

Further, it is known [Krishna Nandan 1973, p. 215] that

$$\log m(\sigma) = \log m(\sigma_1) + \int_{\sigma_1}^{\sigma} \lambda_{N(x)} dx, \; -\infty < \sigma_1 < \sigma < \alpha. \qquad \ldots(3.5)$$

Therefore, since $\limsup_{n\to\infty} \frac{n}{\lambda_n} = D < \infty$, we have for $\sigma > \sigma_1(\epsilon)$ sufficiently close to $\alpha$

$$
\begin{aligned}
N(\sigma)(1 - e^{\sigma-\alpha}) &< (D + \epsilon)\, \lambda_{N(\sigma)} (1 - e^{\sigma-\alpha}) \\
&\leqslant 2(D + \epsilon) \int_{\sigma}^{\sigma+\frac{1}{2}(1-e^{\sigma-\alpha})} \lambda_{N(x)}\, dx \\
&< 2(D + \epsilon) \log m(\sigma + \tfrac{1}{2}(1 - e^{\sigma-\alpha})) \\
&< 2(D + \epsilon) [\log [1 - \exp\{\sigma - \alpha + \tfrac{1}{2}(1 - e^{\sigma-\alpha})\}]^{-1}]^{\rho^*+\epsilon}; \quad \text{by (3.4)}
\end{aligned}
$$

$$
= 2(D + \epsilon)\left[\log\left[(1 - e^{\sigma-\alpha})\left\{1 - \tfrac{1}{2}e^{\sigma-\alpha} - \frac{e^{\sigma-\alpha}(1 - e^{\sigma-\alpha})}{8}\right\}\right]^{-1}\right]^{\rho^*+\epsilon}
$$

$$
< 2(D + \epsilon) [\log \{3(1 - e^{\sigma-\alpha})\}^{-1}]^{\rho^*+\epsilon}.
$$

Thus

$$
N(\sigma)(1 - e^{\sigma-\alpha}) < (D + \epsilon)\, \lambda_{N(\sigma)} (1 - e^{\sigma-\alpha}) < K [\log (1 - e^{\sigma-\alpha})^{-1}]^{\rho^*+\epsilon}. \qquad \ldots(3.6)
$$

Again, it has been shown (Krishna Nandan 1973, p. 216) that if (3.1) holds, then for $\gamma < \beta$

$$
M(\sigma) < m(\sigma)\left[1 + \frac{1 + \gamma}{\gamma} N\left\{\sigma + \frac{1 - e^{\sigma-\alpha}}{N(\sigma)}\right\}\right](1 - e^{\sigma-\alpha})^{-1}.
$$

Putting $\delta = \frac{1 + \gamma}{\gamma}$, and making use of (3.6), we get

$$
\log\log M(\sigma) < \log\log m(\sigma) + \log\left[1 + \frac{2\log(1 - e^{\sigma-\alpha})^{-1} + \log\delta + (\rho^* + \epsilon)\log\log(1 - e^{\sigma-\alpha})^{-1} + o(1)}{\log m(\sigma)}\right].
$$

Dividing by $\log\log(1 - e^{\sigma-\alpha})^{-1}$ and passing to limits, we get

$$
\lim_{\sigma\to\alpha} \begin{matrix}\sup \\ \inf\end{matrix} \frac{\log\log M(\sigma)}{\log\log(1 - e^{\sigma-\alpha})^{-1}} \leqslant \lim_{\sigma\to\alpha} \begin{matrix}\sup \\ \inf\end{matrix} \frac{\log\log m(\sigma)}{\log\log(1 - e^{\sigma-\alpha})^{-1}}. \qquad \ldots(3.7)
$$

The theorem follows by (3.7) and the fact that

$$
m(\sigma) \leqslant M(\sigma).
$$

Next we show that

*Theorem* 3.2 — If $f(s) = \Sigma a_n \exp(s\lambda_n)$ belongs to the class $A_\alpha^*$ and is of logarithmic order $\rho^*$ $(1 \leqslant \rho^* \leqslant \infty)$, then

$$\max\left\{1, \limsup_{n\to\infty} \frac{\log^+\{\alpha\lambda_n + \log|a_n|\}}{\log\log\lambda_n}\right\} = \rho^*. \quad \ldots(3.8)$$

PROOF : Let

$$\max\left\{1, \limsup_{n\to\infty} \frac{\log^+\{\alpha\lambda_n + \log|a_n|\}}{\log\log\lambda_n}\right\} = \mu \quad \ldots(3.9)$$

then $1 \leqslant \mu \leqslant \infty$. First we show that $\mu \leqslant \rho^*$. For this let first, $\rho^* < \infty$, then from (2.2) for given $\epsilon > 0$, $\sigma > \sigma_0 = \sigma_0(\epsilon)$, we have

$$\log M(\sigma) < [\log(1 - e^{\sigma-\alpha})^{-1}]^u, \text{ where } u = \rho^* + \epsilon. \quad \ldots(3.10)$$

Also, we know that

$$\log M(\sigma) \geqslant \log|a_n| + \sigma\lambda_n.$$

This with (3.10) gives, for $\sigma > \sigma_0 = \sigma_0(\epsilon)$

$$\log|a_n| < [\log(1 - e^{\sigma-\alpha})^{-1}]^u - \sigma\lambda_n. \quad \ldots(3.11)$$

Now, choosing $n$ such that

$$\frac{1}{1 - e^{\sigma-\alpha}} = \frac{\lambda_n}{u} \quad \ldots(3.12)$$

it easily follows that

$$\alpha\lambda_n + \log|a_n| < \left(\log\frac{\lambda_n}{u}\right)^u - \lambda_n \log\left(1 - \frac{u}{\lambda_n}\right).$$

Hence

$$\log^+(\alpha\lambda_n + \log|a_n|) < \log\left[\left(\log\frac{\lambda_n}{u}\right)^u - \lambda_n\log\left(1 - \frac{u}{\lambda_n}\right)\right].$$

Dividing both sides by $\log\log\lambda_n$ and proceeding to limit, we have

$$\limsup_{n\to\infty} \frac{\log^+(\alpha\lambda_n + \log|a_n|)}{\log\log\lambda_n} \leqslant u = \rho^* + \epsilon.$$

Since $\epsilon$ can be chosen arbitrarily small, we have

$$\limsup_{n\to\infty} \frac{\log^+(\alpha\lambda_n + \log|a_n|)}{\log\log\lambda_n} \leqslant \rho^*. \quad \ldots(3.13)$$

which obviously holds if $\rho^* = \infty$. Thus $\mu \leqslant \rho^*$.

To complete the proof of theorem, we now show that $\mu \geqslant \rho^*$.

---

$\log^+ x = \log x$, if $x > 1$
$= 0$ otherwise.

Let $\mu < \infty$, then from (3.9) for $\epsilon > 0$, we have for all $n > n_0 = n_0(\epsilon)$

$$\log |a_n| < [(\log \lambda_n)^T - \lambda_n \alpha], \text{ where } T = \mu + \epsilon. \quad \ldots(3.14)$$

Also, $$M(\sigma) \leqslant \sum_{n=1}^{\infty} |a_n| e^{\sigma \lambda_n},$$

therefore, for $\sigma < \alpha$, by using (3.14), we get

$$M(\sigma) < P(n_0) + \sum_{n=n_0+1}^{\infty} \exp[(\log \lambda_n)^T + (\sigma - \alpha)\lambda_n] \quad \ldots(3.15)$$

where $P(n_0)$, the sum of first $n_0$ terms, is bounded.

Now, it is easily seen that if $g(x) = (\log x)^T + rx$, where $r < 0$, then

$$\max_{0 \leqslant x \leqslant \infty} g(x) = \left[\log \frac{T}{-r}\right]^T - T.$$

Thus

$$\exp[(\log \lambda_n)^T + (\sigma - \alpha)\lambda_n] \leqslant \exp\left[\left(\log \frac{T}{\alpha - \sigma}\right)^T - T\right],$$

for all $n$ and hence (3.15) gives

$$M(\sigma) < P(n_0) + N \exp\left[\left(\log \frac{T}{\alpha - \sigma}\right)^T - T\right]$$

$$+ \sum_{n=N+1}^{\infty} \exp[(\log \lambda_n)^T + (\sigma - \alpha)\lambda_n]$$

where $$N = (D + \epsilon) \frac{2}{\alpha - \sigma} \left[\log \frac{2}{\alpha - \sigma}\right]^T.$$

If $\sigma$ is sufficiently close to $\alpha$, and $n > N$, then by using (1.2), we have

$$\sum_{n=N+1}^{\infty} \exp[(\log \lambda_n)^T + (\sigma - \alpha)\lambda_n] < \sum_{n=N+1}^{\infty} \exp\left\{(\sigma - \alpha)\frac{\lambda_n}{2}\right\}$$

$$< \sum_{n=N+1}^{\infty} \exp\left\{\frac{\sigma - \alpha}{2} \frac{n}{D + \epsilon}\right\}$$

$$= \frac{\exp\left\{\frac{\sigma - \alpha}{2} \cdot \frac{N + 1}{D + \epsilon}\right\}}{1 - \exp\left(\frac{\sigma - \alpha}{2(D + \epsilon)}\right)}.$$

Since

$$\lim_{\sigma\to\alpha} \frac{\exp\left\{\frac{(\sigma-\alpha)}{2}\frac{N+1}{D+\epsilon}\right\}}{1-\exp\left[\frac{\sigma-\alpha}{2(D+\epsilon)}\right]} = \lim_{\sigma\to\alpha} \frac{\exp\left[-\left\{\log\frac{2}{\alpha-\sigma}\right\}^T\right]}{-\frac{(\sigma-\alpha)}{2(D+\epsilon)}[1+o(\sigma-\alpha)]}$$

$$= 0, \text{ as } \sigma\to\alpha (T>1)$$

we have

$$\sum_{n=N+1}^{\infty} \exp\{(\log\lambda_n)^T + (\sigma-\alpha)\lambda_n = o(1), \text{ as } \sigma\to\alpha.$$

Thus, $$M(\sigma) \leqslant P(n_0) + N\exp\left(\left[\log\frac{T}{\alpha-\sigma}\right]^T - T\right) + o(1)$$

or, $$\log\log M(\sigma) \leqslant T\log\left(\log\frac{T}{\alpha-\sigma}\right)(1-o(1)) + o(1).$$

Dividing by $\log\log(1-e^{\sigma-\alpha})^{-1}$ on both sides and proceeding to limits, we get

$$\rho^* \leqslant T = \mu + \epsilon.$$

Since $\epsilon$ can be made arbitrarily small, we obtain

$$\rho^* \leqslant \mu$$

which holds if $\mu = \infty$. Hence the theorem.

In the following theorem we construct an example of a function in which $\rho^* > \lambda^*$ and show that the result analogous to (3.8) for $\lambda^*$ does not always hold.

*Theorem* 3.3 — There exists an analytic function $f(s) = \sum_1^{\infty} a_n \exp(s\lambda_n) \in A_\alpha^*$ such that (i) $a_n > 0$, (ii) $\liminf_{n\to\infty}(\lambda_{n+1}-\lambda_n) = \beta > 0$, and (iii) $\rho^* > \lambda^*$.

To prove the theorem first we prove the following lemma :

*Lemma* — If $f(s) = \sum_{n=1}^{\infty} a_n \exp(s\lambda_n)$ belongs to the class $A_\alpha^*$ having lower logarithmic order $\lambda^*$ $(1 \leqslant \lambda^* \leqslant \infty)$, and if $\{n_k\}_1^\infty$ is an increasing sequence of positive integers, then

$$\lambda^* \geqslant \liminf_{k\to\infty} \frac{\log^+(\alpha\lambda_{n_k} + \log|a_{n_k}|)}{\log\log\lambda_{n_{k+1}}}. \qquad \ldots(3.16)$$

PROOF : Let first

$$\liminf_{k\to\infty} \frac{\log^+(\alpha\lambda_{n_k} + \log|a_{n_k}|)}{\log\log\lambda_{n_{k+1}}} = \theta \quad (0 \leqslant \theta < \infty).$$

Then, for any given $\epsilon > 0$, we can find a fixed integer $N = N(\epsilon)$ such that for $k > N$

$$\log^{+}(\alpha\lambda_{n_k} + \log | a_{n_k} | ) > \log (\log \lambda_{n_{k+1}})^{\theta-\epsilon}$$

or
$$\log | a_{n_k} | > (\log \lambda_{n_{k+1}})^{\theta-\epsilon} - \alpha\lambda_{n_k}.$$

Choose
$$\alpha - \sigma_k = \frac{\theta - \epsilon}{\lambda_{n_k}}; k = 1, 2, 3 \ldots.$$

If $k > N$ and $\sigma_k \leqslant \sigma \leqslant \sigma_{k+1}$, then by using Cauchy's inequality, we get

$$\begin{aligned}\log M(\sigma) &\geqslant \log | a_{n_k} | + \lambda_{n_k} \sigma \\ &> (\log \lambda_{n_{k+1}})^{\theta-\epsilon} - (\alpha - \sigma_k) \lambda_{n_k} \\ &= [\log (\alpha - \sigma_{k+1})^{-1}]^{\theta-\epsilon} + O(1)\end{aligned}$$

or
$$\begin{aligned}\log \log M(\sigma) &\geqslant (\theta - \epsilon) \log \log (\alpha - \sigma_{k+1})^{-1} + o(1). \\ &\geqslant (\theta - \epsilon) \log \log (\alpha - \sigma)^{-1} + o(1).\end{aligned}$$

From this we easily have

$$\lambda^{*} = \liminf_{\sigma \to \alpha} \frac{\log \log M(\sigma)}{\log \log (1 - e^{\sigma-\alpha})^{-1}} \geqslant \theta.$$

If $\theta = \infty$, it can be easily seen that $\lambda^{*} = \infty$.

This proves the Lemma.

PROOF OF THEOREM 3.3 : Let

$$f(s) = \sum_{n=1}^{\infty} \exp \{(\log \lambda_n)^2 + s\lambda_n\} \qquad \ldots(3.17)$$

where
$$\log \lambda_1 = 2, \log \lambda_{n+1} = (\log \lambda_n)^2.$$

Then $f(s)$ is analytic in the half-plane $\operatorname{Re} s < \alpha = 0$. Obviously $a_n > 0$ and

$$\liminf_{n \to \infty} (\lambda_{n+1} - \lambda_n) > 0.$$

Further, let $\sigma_n = \dfrac{(\log \lambda_n)^2 - (\log \lambda_{n+1})^2}{\lambda_{n+1} - \lambda_n}$, then it is seen that $\sigma_n$ is a non-decreasing function of $n$.

If $n$th term is the maximum term, then

$$\log m(\sigma, f) = (\log \lambda_n)^2 + \sigma\lambda_n$$

and
$$\lambda_{N(\sigma)} = \lambda_n, \text{ for } \sigma_{n-1} \leqslant \sigma < \sigma_n.$$

Now for $\sigma = \sigma_{n-1}$

$$\frac{\log\log m(\sigma, f)}{\log\log(1-e^{\sigma})^{-1}} \sim \frac{\log[(\log\lambda_n)^2 + \sigma_{n-1}\lambda_n]}{\log\log[-\sigma_{n-1}]^{-1}}$$

$$\sim \frac{\log\left[\dfrac{\lambda_n(\log\lambda_{n-1})^2[1+o(1)]}{\lambda_n[1+o(1)]}\right]}{\log\log\left[\dfrac{\lambda_n[1+o(1)]}{(\log\lambda_n)^2[1+o(1)]}\right]}$$

$$\sim \frac{\log(\log\lambda_{n-1})^2}{\log\log\lambda_n} \to 1$$

since $\log\lambda_n = (\log\lambda_{n-1})^2$.

Also, by using (3.16) we have $\lambda^* \geqslant 1$, hence it follows that

$$\lambda^* = \liminf_{\sigma\to 0} \frac{\log\log m(\sigma, f)}{\log\log(1-e^{\sigma})^{-1}} = 1.$$

Further, it is seen that

$$\liminf_{n\to\infty} \frac{\log^+[\alpha\lambda_n + \log|a_n|]}{\log\log\lambda_n} = \limsup_{n\to\infty} \frac{\log^+[\alpha\lambda_n + \log|a_n|]}{\log\log\lambda_n} = 2.$$

Hence by (3.8) $\rho^* = 2$. Thus $\rho^* > \lambda^*$ and the relation analogous to (3.8) for $\lambda^*$ does not hold in this case.

Next, we show that

§4. *Theorem* 4.1 — If $f(s) = \sum_{n=1}^{\infty} a_n \exp(s\lambda_n)$ belongs to the class $A_\alpha^*$ with logarithmic order $\rho^*(1 < \rho^* < \infty)$ and satisfies (3.1), then

$$\rho^* - 1 \leqslant \limsup_{\sigma\to\alpha} \frac{\log[\lambda_{N(\sigma)}(1-e^{\sigma-\alpha})]}{\log\log(1-e^{\sigma-\alpha})^{-1}} \leqslant \rho^*. \qquad \ldots(4.1)$$

PROOF: Let

$$\limsup_{\sigma\to\alpha} \frac{\log[\lambda_{N(\sigma)}(1-e^{-\alpha\sigma})]}{\log\log(1-e^{\sigma-\alpha})^{-1}} = \theta.$$

Then for $0 < \theta < \theta_1 < \infty$, there exists a $\sigma_0$ such that for $\sigma_0 < \sigma < \alpha$,

$$\lambda_{N(\sigma)}(1-e^{\sigma-\alpha}) < [\log(1-e^{\sigma-\alpha})^{-1}]^{\theta_1}.$$

But by (3.5)

$$\log m(\sigma) = \log m(\sigma_0) + \int_{\sigma_0}^{\sigma} \lambda_{N(x)}\, dx$$

where $-\infty < \sigma_0 < \sigma < \alpha$.

Therefore for $\sigma$ sufficiently close to $\alpha$

$$\log m(\sigma) < \log m(\sigma_0) + \int_{\sigma_0}^{\sigma} \frac{[\log (\alpha - x)^{-1}]^{\theta_1}}{\alpha - x} dx$$

$$= \frac{[\log (\alpha - \sigma)^{-1}]^{\theta_1+1}}{\theta_1 + 1} + O(1)$$

or, $$\log \log m(\sigma) < (\theta_1 + 1) \log \log (\alpha - \sigma)^{-1} + O(1). \quad \ldots(4.2)$$

Dividing by $\log \log (1 - e^{\sigma-\alpha})^{-1}$ and proceeding to limits, we have

$$\rho^* \leqslant \theta_1 + 1.$$

Since it is true for every $\theta_1 > \theta$, so

$$\rho^* \leqslant \theta + 1. \quad \ldots(4.3)$$

If $\theta = 0$ then $\rho^* = 1$.

Further, by (3.6)

$$\lambda_{N(\sigma)} (1 - e^{\sigma-\alpha}) < K [\log (1 - e^{\sigma-\alpha})^{-1}]^{\rho^*+\epsilon}$$

or, $$\log [\lambda_{N(\sigma)} (1 - e^{\sigma-\alpha})] < (\rho^* + \epsilon) \log \log (1 - e^{\sigma-\alpha})^{-1} + O(1).$$

This gives

$$\rho^* \geqslant \limsup_{\sigma \to \alpha} \frac{\log [\lambda_{N(\sigma)} (1 - e^{\sigma-\alpha})]}{\log \log (1 - e^{\sigma-\alpha})^{-1}} = \theta. \quad \ldots(4.4)$$

Combining (4.3) and (4.4), we get (4.1).

*Remark* : The relation (4.1) is best possible as is seen by the following examples.

*Example* 4.1 — Let $f(s) = \sum_{n=1}^{\infty} n \exp (sn)$, then $f(s)$ is analytic in the half-plane Re $s < 0$.

If $n$th term is the maximum term, then

$$\log m(\sigma, f) = \log n + \sigma n$$

and $$\lambda_{N(\sigma)} = n \text{ for } \sigma_{n-1} \leqslant \sigma < \sigma_n,$$

where $$\sigma_{n-1} = \log (n - 1) - \log n$$

$$= \log \left(1 - \frac{1}{n}\right).$$

Now for $\sigma = \sigma_{n-1}$

$$\frac{\log [\lambda_{N(\sigma)} (1 - e^{\sigma})]}{\log \log (1 - e^{\sigma})^{-1}} \sim \frac{\log\left[- n \log\left(1 - \frac{1}{n}\right)\right]}{\log \log\left[- \log\left(1 - \frac{1}{n}\right)\right]^{-1}}$$

$$\sim \frac{\log\left[1 + \frac{1}{2n} \cdots\right]}{\log \log\left[\frac{1}{n} + \frac{1}{2n^2} \cdots\right]^{-1}} \to 0$$

and since, $\dfrac{\log [\lambda_{N(\sigma_{n-1})} (1 - e^{\sigma_{n-1}}]}{\log \log (1 - e^{\sigma_{n-1}})^{-1}} \geqslant \dfrac{\log[\lambda_{N(\sigma)} (1 - e^{\sigma})]}{\log \log (1 - e^{\sigma})^{-1}}$

so, $\displaystyle \limsup_{\sigma \to 0} \frac{\log [\lambda_{N(\sigma)} (1 - e^{\sigma})]}{\log \log (1 - e^{\sigma})^{-1}} = 0.$

Further, by (3.8) $\rho^* = 1$.

Thus, $\displaystyle \limsup_{\sigma \to 0} \frac{\log [\lambda_{N(\sigma)} (1 - e^{\sigma})]}{\log \log (1 - e^{\sigma})^{-1}} \doteq \rho^* - 1$ in this case.

*Example* 4.2 — Let $f(s)$ represent an analytic function given by (3.17), then it is shown that $\rho^* = 2$ and $\lambda^* = 1$.

Further, $\lambda_{N(\sigma)} = \lambda_n$, for $\sigma_{n-1} \leqslant \sigma < \sigma_n$, and

$$\sigma_n = \frac{(\log \lambda_n)^2 - (\log \lambda_{n+1})^2}{\lambda_{n+1} - \lambda_n}.$$

Therefore, for $\sigma = \sigma_{n-1}$,

$$\frac{\log [\lambda_{N(\sigma)} (1 - e^{\sigma})]}{\log \log (1 - e^{\sigma})^{-1}} \sim \frac{\log [\lambda_n(- \sigma)]}{\log \log [- \sigma]^{-1}}$$

$$\sim \frac{\log [- \lambda_n(\sigma_{n-1})]}{\log \log [- \sigma_{n-1}]^{-1}}$$

$$\sim \frac{\log\left[\frac{\lambda_n(\log \lambda_n)^2 [1 + o(1)]}{\lambda_n [1 + o(1)]}\right]}{\log \log\left[\frac{\lambda_n [1 + o(1)]}{(\log \lambda_n)^2 [1 + o(1)]}\right]}$$

$$\sim \frac{\log (\log \lambda_n)^2}{\log \log \lambda_n} \to 2.$$

Further, we have

$$\frac{\log[\lambda_{N(\sigma_{n-1})} (1 - e^{\sigma_{n-1}})]}{\log \log (1 - e^{\sigma_{n-1}})^{-1}} \geqslant \frac{\log [\lambda_{N(\sigma)} (1 - e^{\sigma})]}{\log \log (1 - e^{\sigma})^{-1}}.$$

Hence it follows that

$$\limsup_{\sigma \to 0} \frac{\log [\lambda_{N(\sigma)} (1 - e^{\sigma})]}{\log \log (1 - e^{\sigma})^{-1}} = 2.$$

Thus, $$\limsup_{\sigma \to 0} \frac{\log \lambda_{N(\sigma)} (1 - e^{\sigma})}{\log \log (1 - e^{\sigma})^{-1}} = 2 = \rho^* \text{ in the case.}$$

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