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PROJECTIVE MOTION IN AN n-DIMENSIONAL SPECIAL KAWAGUCHI SPACE

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The object of the present paper is to introduce the idea of infinitesimal projective transformation in an n-dimensional special Kawaguchi space and to establish some theorems on projective motion and curvature collineation.

1. INTRODUCTION

Kawaguchi (1938) has defined a space with metric function $F = A_i x''^i + B$, such that $ds = F^{1/p} dt$. In view of this special type of metric for which $p \neq 3/2, 0$; the space is called an n-dimensional special Kawaguchi space $K_n^{(1)}$ $(n > 2)$. Here A_i and B are the differentiable functions of x^i and x'^i and are homogeneous functions of degree $p - 2$ and p respectively. We shall use some of the ideas, notations and results given by Kawaguchi without explanation.

We consider a contravariant vector field $X^i(x, x')$ whose covariant derivative is given by

$$\nabla_j X^i = \partial_j X^i - \partial'_k X^i \Gamma^k_{(j)} + \Gamma^i_{(k)(j)} X^k \qquad ...(1.1)$$

where $\partial_j = \dfrac{\partial}{\partial x^j}$, $\partial'_j = \dfrac{\partial}{\partial x'^j}$ and the connection parameter $\Gamma^i(x, x')$ is positively homogeneous of degree 2 with respect to x'^i. The commutation formulae are given by

$$(\nabla_j \nabla_k - \nabla_k \nabla_j) X^i = - R_{jkl}^{\cdot\cdot\cdot i} X^l + K_{jk}^{\cdot\cdot l} \nabla'_l X^i \qquad ...(1.2)$$

and

$$(\nabla_j \nabla'_k - \nabla'_k \nabla_j) X^i = - B_{jkl}^{\cdot\cdot\cdot i} X^l \qquad ...(1.3)$$

where

$$R_{jkl}^{\cdot\cdot i} = 2\{\partial_{[k} \Gamma^i_{(j)](l)} + \Gamma^h_{(l)[(j)} \Gamma^i_{(k)](h)} + \Gamma^h_{[(j)} \Gamma^i_{(k)](l)(h)}\} \qquad ...(1.4)$$

$$K_{jk}^{\cdot\cdot i} = 2\{\partial_{[k} \Gamma^i_{(j)]} + \Gamma^h_{[(j)} \Gamma^i_{(k)](h)}\} \qquad ...(1.5)$$

and

$$B_{jkl}^{\cdots i} = \Gamma^i_{(j)(k)(l)}. \qquad \ldots(1.6)$$

These curvature tensors satisfy the following identities:

$$\left.\begin{aligned}
&\text{(a)} \quad R_{jkl}^{\cdots i} + R_{klj}^{\cdots i} + R_{ljk}^{\cdots i} = 0 \\
&\text{(b)} \quad R_{jkl}^{\cdots i} = - R_{kjl}^{\cdots i} \\
&\text{(c)} \quad K_{jk}^{\cdot\cdot i} = R_{jkl}^{\cdots i}\, x'^l \\
&\text{(d)} \quad R_{jkl}^{\cdots i} = K_{jk(l)}^{\cdot\cdot i} = \nabla'_l K_{jk}^{\cdot\cdot i}.
\end{aligned}\right\} \qquad \ldots(1.7)$$

2. Infinitesimal Transformation

Let us consider the infinitesimal point transformation

$$\bar{x}^i = x^i + V^i(x)\, d\tau, \qquad \ldots(2.1)$$

where $\bar{x}^i$ and x^i are the coordinates in a special Kawaguchi space and V^i the contravariant components of vector defining the transformation. This vector is function of x^i only. Lie derivative of any vector X^i is given by Yano (1957)

$$\underset{v}{\mathcal{L}}\, X^i = \frac{d^v X^i - d^m X^i}{d\tau} \qquad \ldots(2.2)$$

where the differentials $d^v X^i$ and $d^m X^i$ are obtained in view of point transformation and coordinate transformation of the vector X^i. Hence, the Lie derivative of $X^i(x, x')$ is given by

$$\underset{v}{\mathcal{L}}\, X^i = V^j \partial_j X^i + \partial'_k X^i \partial_j V^k x'^j - X^j \partial_j V^i. \qquad \ldots(2.3)$$

The Lie derivative of $\Gamma^i_{(j)(k)}$ (Yano 1957) is given by

$$\underset{v}{\mathcal{L}}\, \Gamma^i_{(j)(k)} = V^h \partial_h \Gamma^i_{(j)(k)} + \Gamma^i_{(j)(k)(r)}\, \partial_j V^r x'^h$$

$$+ \partial_j \partial_k V^i - \partial_r V^i\, \Gamma^r_{(j)(k)} + \partial_k V^r\, \Gamma^i_{(j)(r)} + \partial_j V^r\, \Gamma^i_{(r)(k)}. \qquad \ldots(2.4)$$

Using the relations (2.3), (2.4) and (1.1), we obtain

$$\underset{v}{\mathcal{L}}\, X^i = V^j \nabla_j X^i + \partial'_k X^i \nabla_j V^k x'^j - X^j \nabla_j V^i \qquad \ldots(2.5)$$

and

$$\underset{v}{\mathcal{L}}\, \Gamma^i_{(j)(k)} = \nabla_j \nabla_k V^i + R_{jkl}^{\cdots i}\, V^l + \partial'_r \Gamma^i_{(j)(k)} \nabla_h V^r x'^h. \qquad \ldots(2.6)$$

With the help of (2.6), we get

$$\nabla_l (\underset{v}{\mathcal{L}} \Gamma^i_{(j)(k)}) - \nabla_k (\underset{v}{\mathcal{L}} \Gamma^i_{(j)(l)})$$

$$= \underset{v}{\mathcal{L}} R^{\cdots i}_{jkl} + 2\partial'_r \Gamma^i_{(j)[(k)} \underset{v}{\mathcal{L}} \Gamma^r_{(l)](h)} x'^h. \qquad \ldots(2.7)$$

3. Projective Motion and Curvature Collineation in $K_n^{(1)}$

Definition 3.1 — If an infinitesimal transformation (2.1) transforms the system of paths into the same system, then the infinitesimal transformation is said to be projective motion. The projective motion in special Kawaguchi space (Pande and Singh 1974) is given by

$$\underset{v}{\mathcal{L}} \Gamma^i_{(j)(k)} = \delta^i_k \alpha_{(j)} + \delta^i_j \alpha_{(k)} + \alpha_{(j)(k)} x'^i. \qquad \ldots(3.1)$$

From (3.1), we get

$$\underset{v}{\mathcal{L}} \Gamma^i_{(j)} = \alpha_{(j)} x'^i + \alpha \delta^i_j \ . \qquad \ldots(3.2)$$

and

$$\underset{v}{\mathcal{L}} \Gamma^i_{(j)(k)(l)} = \alpha_{(j)(k)} \delta^i_l + \alpha_{(k)(l)} \delta^i_j + \alpha_{(l)(j)} \delta^i_k + \alpha_{(j)(k)(l)} x'^i \qquad \ldots(3.3)$$

where

$$\underset{v}{\mathcal{L}} x'^i = 0.$$

Definition 3.2 — An n-dimensional special Kawaguchi space is said to admit R-curvature collineation if there exists an infinitesimal transformation (2.1) for which the following relation is true

$$\underset{v}{\mathcal{L}} R^{\cdots i}_{jkl} = 0. \qquad \ldots(3.4)$$

Equation (3.1) reduces to

$$\underset{v}{\mathcal{L}} S^i_{jk} = 0$$

where

$$S^i_{jk} = \Gamma^i_{(j)(k)} - \frac{1}{n+1} \Gamma^r_{(r)(k)} \delta^i_j - \frac{1}{n+1} \Gamma^r_{(r)(j)} \delta^i_k - \frac{1}{n+1} \Gamma^r_{(r)(j)(k)} x'^i. \qquad \ldots(3.5)$$

Thus we have the following :

Theorem 3.1 — A $K_n^{(1)}$ admits the projective motion provided there exists a vector V^i for which

$$\underset{v}{\mathcal{L}}\, S^i_{jk} = 0.$$

In view of eqns. (3.1), (3.2), (3.3) and (1.4), we get

$$\underset{v}{\mathcal{L}}\, R^{\cdots i}_{jkl} = \delta^i_j \nabla_k \alpha_{(l)} - \delta^i_k \nabla_j \alpha_{(l)} + \delta^i_l (\nabla_k \alpha_{(j)} - \nabla_j \alpha_{(k)})$$
$$+ x'^i (\nabla_k \alpha_{(l)(j)} - \nabla_j \alpha_{(l)(k)}). \qquad \ldots(3.6)$$

Multiplying eqn. (3.6) by x'^1 and x'^j successively, we get

$$\underset{v}{\mathcal{L}}\, K^{\cdot\cdot i}_{jk} = \delta^i_j \nabla_k \alpha - \delta^i_k \nabla_j \alpha + x'^i (\nabla_k \alpha_{(j)} - \nabla_j \alpha_{(k)}) \qquad \ldots(3.7)$$

and

$$\underset{v}{\mathcal{L}}\, H^i_k = 2x'^i \nabla_k \alpha - \delta^i_k x'^j \nabla_j \alpha - x'^j \nabla_j \alpha_{(k)} x'^i. \qquad \ldots(3.8)$$

Contracting (3.8) with respect to indices i and k, we get

$$\underset{v}{\mathcal{L}}\, H = - x'^i \nabla_i \alpha. \qquad \ldots(3.9)$$

The Lie derivatives of tensors P_{kl} and Q_{kl} (Kawaguchi 1964) are given by

$$\underset{v}{\mathcal{L}}\, P_{kl} = - \nabla_k \alpha_{(l)} + \frac{1}{n-1} x'^m \nabla_m \alpha_{(l)(k)} \qquad \ldots(3.10)$$

$$\underset{v}{\mathcal{L}}\, Q_{kl} = - \partial'_l \nabla_k \alpha. \qquad \ldots(3.11)$$

On summarizing the above results, we get the following theorem:

Theorem 3.2 — The Lie derivative of the tensors which are not projective invariant, do not vanish identically.

In view of eqns. (3.1), (3.2), (3.3), (3.10) and (3.11), the Lie derivative of the projective invariants W^i_j, W^i_{jk}, W^i_{jkl} and U^i_{jkl} (Kawaguchi 1964) are zero. Thus we get the following.

Theorem 3.3 — The Lie derivative of the projective invariants W^i_j, W^i_{jk}, W^i_{jkl} and U^i_{jkl} vanish identically.

Corollary 3.1 — Every projective motion is projective curvature collineation in $K_n^{(1)}$.

4. Concrete form of Projective Motion

Let us introduce a quantity by

$$L_{jk}^{i} \overset{def}{=} \{\delta_j^i \; P_{kl} - \delta_k^i \; P_{jl} - \delta_l^i \; (P_{jk} - P_{kj})\} \, x'^l \qquad ...(4.1)$$

which is skew-symmetric with respect to its lower indices and positive homogeneous of degree one with respect to x'^i. The projective curvature tensor $W_{jk}^{i}(x, x')$ (Kawaguchi 1964) is defined by

$$W_{jk}^{i} = K_{jk}^{\cdot\cdot i} + L_{jk}^{i}. \qquad ...(4.2)$$

An n-dimensional special Kawaguchi space with symmetric connection $\Gamma_{(j)(k)}^{i}$ is said to be recurrent (Takano 1962) if the covariant derivative of the curvature tensor $R_{jkl}^{\cdots i}$ satisfies the relation

$$\nabla_m R_{jkl}^{\cdots i} = K_m R_{jkl}^{\cdots i}, \; R_{jkl}^{\cdots i} \neq 0 \qquad ...(4.3)$$

where K_m is a non null recurrence vector positive homogeneous of degree zero with respect to its directional argument. This recurrent space is denoted by R - $K_n^{(1)}$ space.

In such a specialized $K_n^{(1)}$, let us try to discuss the existence of projective motion (3.2). We know that for projective motion $\underset{v}{\mathcal{L}} W_{jk}^{i} = 0$, therefore, we shall find an important property, on W_{jk}^{i}, holding in R - $K_n^{(1)}$ admitting projective motion.

In view of eqn. (4.3) the tensor P_{kl} (Kawaguchi 1964) is recurrent for the same recurrent vector K_m and hence L_{jk}^{i} is recurrent. With the help of eqns. (1.7c), (4.1), (4.2) and (4.3), we get

$$\nabla_m W_{jk}^{i} = K_m W_{jk}^{i}. \qquad ...(4.4)$$

Applying the operator Lie derivative $(\underset{v}{\mathcal{L}})$ throughout eqn. (4.4) and using the fact that $\underset{v}{\mathcal{L}}\, W^i_{jk} = 0$, we get

$$\underset{v}{\mathcal{L}}\,(\nabla_m\, W^i_{jk}) = (\underset{v}{\mathcal{L}}\, K_m)\, W^i_{jk}. \qquad \ldots(4.5)$$

Using the formula (2.7) for $W^i_{jk}\,(x, x')$ and putting the values of Lie derivatives of connection parameter, we get

$$(\underset{v}{\mathcal{L}}\, K_m)\; W^i_{jk} = -\, 2\alpha_{(m)}\, W^i_{jk} - \alpha_{(j)}\, W^i_{mk} - \alpha_{(k)}\, W^i_{jm}$$
$$+\, \alpha_{(l)}\, \delta^i_m\, W^l_{jk} + \alpha_{(l)(m)}\, x'^l\, W^l_{jk} - \alpha_{(m)(j)}\, x'^l\, W^i_{lk}$$
$$-\, \alpha_{(m)(k)}\, x'^l\, W^i_{jl} - \alpha \dot\partial_m\, W^i_{jk}. \qquad \ldots(4.6)$$

Contracting eqn. (4.6) with respect to indices i and m, we get

$$(\underset{v}{\mathcal{L}}\, K_m)\; W^m_{jk} = (n-2)\, \alpha_{(l)}\, W^l_{jk} + \alpha_{(i)(k)}\, W^i_j - \alpha_{(i)(j)}\, W^i_k\,. \qquad \ldots(4.7)$$

Multiplying both the sides of (4.6) by $\alpha_{(i)}$ and summing up over i we may obtain

$$(\underset{v}{\mathcal{L}}\, K_m)\, \alpha_{(i)}\, W^i_{jk} = -\, \alpha_{(m)}\, \alpha_{(i)}\, W^i_{jk} - \alpha_{(i)}\, \alpha_{(j)}\, W^i_{mk}$$
$$-\, \alpha_{(i)}\, \alpha_{(k)}\, W^i_{jm} + \alpha\, \alpha_{(l)(m)}\, W^l_{jk} - x'^l\, \alpha_{(m)(j)}\, \alpha_{(i)}\, W^i_{lk}$$
$$-\, x'^l\, \alpha_{(m)(k)}\, \alpha_{(i)}\, W^i_{jl} - \alpha\alpha_{(i)}\, \dot\partial_m\, W^i_{jk}. \qquad \ldots(4.8)$$

In view of eqn. (4.7) for $n \geqslant 3$, eqn. (4.8) reduces to

$$\underset{v}{\mathcal{L}}\, K_m\{\underset{v}{\mathcal{L}}\, K_h\, W^h_{jk} - T^h_{hjk}\} = -\, \alpha_{(m)}\{\underset{v}{\mathcal{L}}\, K_h\, W^h_{jk} - T^h_{hjk}\}$$
$$-\, \alpha_{(j)}\{\underset{v}{\mathcal{L}}\, K_h\, W^h_{mk} - T^h_{hmk}\} - \alpha_{(k)}\{\underset{v}{\mathcal{L}}\, K_h\, W^h_{jm} - T^h_{hjm}\} + \alpha_{(i)}\, T^i_{mjk}$$
$$\ldots(4.9)$$

where

$$T^i_{mjk} = x'^i\, \alpha_{(l)(m)}\, W^l_{jk} - x'^l\, \alpha_{(m)(j)}\, W^i_{lk} - x'^l\, \alpha_{(m)(k)}\, W^i_{jl} - \alpha\dot\partial_m\, W^i_{jk}$$
$$\ldots(4.10)$$

which is positive homogeneous of degree one with respect to x'^i and skew-symmetric with respect to indices j and k. $T^i_{mjk}(x, x')$ satisfies the following identities:

$$\left.\begin{array}{ll}\text{(a)} & T^h_{hjk} = \alpha_{(i)(k)} W^i_j - \alpha_{(i)(j)} W^i_k \\ \text{(b)} & T^h_{hjk}\, x'^j = 0 \\ \text{(c)} & T^h_{hjk}\, x'^k = 0.\end{array}\right\} \qquad \ldots(4.11)$$

Equation (4.9) is rewritten as

$$(\underset{v}{\mathcal{L}} K_h)\{(\underset{v}{\mathcal{L}} K_m) W^h_{jk} + \alpha_{(m)} W^h_{jk} + \alpha_{(j)} W^h_{mk} + \alpha_{(k)} W^h_{jm}\}$$

$$= \alpha_{(i)} T^i_{mjk} + (\underset{v}{\mathcal{L}} K_m) T^h_{hjk} + \alpha_{(m)} T^h_{hjk}$$

$$+ \alpha_{(j)} T^h_{hmk} + \alpha_{(k)} T^h_{hjm}. \qquad \ldots(4.12)$$

Thus we get the following:

Theorem 4.1 — If an n-dimensional recurrent $K^{(1)}_n (n \geqslant 3)$ admits an infinitesimal projective motion (3.2), then the recurrence vector K_m satisfies eqn. (4.12).

Using eqn. (4.7) in (4.12), we get

$$\{(\underset{v}{\mathcal{L}} K_m) - (n-2)\,\alpha_{(m)}\}(\underset{v}{\mathcal{L}} K_h) W^h_{jk} = (\underset{v}{\mathcal{L}} K_m) T^h_{hjk}$$

$$+ (n-2)\{\alpha_{(i)} T^i_{mjk} + (\underset{v}{\mathcal{L}} K_m) T^h_{hjk} + \alpha_{(m)} T^h_{hjk}$$

$$+ \alpha_{(j)} T^h_{hmk} + \alpha_{(k)} T^h_{hjm}\}. \qquad \ldots(4.13)$$

If we put

$$(\underset{v}{\mathcal{L}} K_m) T^h_{hjk} + (n-2)\{\alpha_{(i)} T^i_{mjk} + (\underset{v}{\mathcal{L}} K_m) T^h_{hjk} + \alpha_{(m)} T^h_{hjk}$$

$$+ \alpha_{(j)} T^h_{hmk} + \alpha_{(k)} T^h_{hjm}\} = 0. \qquad \ldots(4.14)$$

Equation (4.13) gives

$$\alpha_{(m)} = \frac{1}{n-2}(\underset{v}{\mathcal{L}} K_m). \qquad \ldots(4.15)$$

Accordingly, we get the following:

Theorem 4.2 — An n-dimensional recurrent $K^{(1)}_n$ $(n \geqslant 3)$ admits an infinitesimal projective motion of the form

$$\bar{x}^i = x^i + v^i(x)\,d\tau,\ \underset{v}{\mathcal{L}}\,\Gamma^i_{(j)(m)} = \alpha_{(j)}\delta^i_m + \alpha_{(m)}\delta^i_j + \alpha_{(j)(m)}\,x'^i$$

$$\alpha = \frac{1}{n-2}(\underset{v}{\mathcal{L}} K_m)\,x'^m.$$

if and only if the relation (4.14) is true.

Next, let us examine a case where $(\underset{v}{\mathcal{L}} K_j)$ denotes a parallel vector i.e. $\nabla_k (\underset{v}{\mathcal{L}} K_j) = 0$. Applying the operator ∇ throughout eqn. (4.15), we get

$$\nabla_k \alpha_{(m)} = \frac{1}{n-2} \nabla_k (\underset{v}{\mathcal{L}} K_m) = 0. \qquad ...(4.16)$$

Using eqns. (3.10) and (4.16), we get

$$\underset{v}{\mathcal{L}} P_{kl} \, x'^l = - \nabla_k \alpha = 0 \qquad ...(4.17)$$

$$\underset{v}{\mathcal{L}} (P_{kl} - P_{lk}) = \nabla_l \alpha_{(k)} - \nabla_k \alpha_{(l)} = 0 \qquad ...(4.18)$$

Operating $\underset{v}{\mathcal{L}}$ to both the sides of eqn. (4.1), we get

$\underset{v}{\mathcal{L}} L^i_{jk} = 0$ and hence in view of (4.2) $\underset{v}{\mathcal{L}} K_{jk}^{\cdot\cdot i} = 0$. Therefore, we get the following:

Theorem 4.3 — The projective motion becomes $K_{jk}^{\cdot\cdot i}$ — curvature collineation if $\underset{v}{\mathcal{L}} K_j$ denotes a parallel vector and eqn. (4.14) is true.

We know that a projective motion becomes an affine motion if $\alpha = 0$, $\alpha_{(j)} = 0$ and $\alpha_{(j)(k)} = 0$. Thus from (4.15), we get the following:

Theorem 4.4 — A projective motion admitted in R - $K_n^{(1)}$ becomes an affine motion if $\underset{v}{\mathcal{L}} K_j = 0$, while (4.14) is true.

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