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ON THE ABSOLUTE RIESZ SUMMABILITY OF LEGENDRE SERIES

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The authors have investigated an application of absolute Riesz summability method of general type of Riesz means and order one to Legendre series and its related series, which leads to a number of interesting results.

1. DEFINITIONS AND NOTATIONS

Let $\lambda = \lambda(w)$ be a continuous, differentiable and monotonic increasing function of w and let it tend to infinity with w. Suppose that $\sum_{n=1}^{\infty} a_n$ be a given infinite series, then

$$\sum_{n=1}^{\infty} a_n \in | R, \lambda(w), r | \quad (r > 0)$$

if
$$\int_A^{\infty} \frac{\lambda^{(1)}(w)}{(\lambda(w))^{r+1}} \left| \sum_{n \leqslant w} \{\lambda(w) - \lambda(n)\}^{r-1} \lambda(n)\, a_n \right| dw < \infty$$

where A is a suitable finite number (Obrechkoff, 1928) and

$$\lambda^{(1)}(w) = \frac{d}{dw} \lambda(w).$$

Let f be a Lebesgue measurable function over the linear interval $[-1, +1]$. Then its Legendre series at $x \in [-1, +1]$ is given by

$$\sum_{n=1}^{\infty} a_n P_n(x) \qquad \ldots(1.1)$$

where

$$a_n = (n + \tfrac{1}{2}) \int_{-1}^{+1} f(y)\, P_n(y)\, dy \qquad \ldots(1.2)$$

and $P_n(x)$ denotes the nth Legendre polynomial.

Among other properties of $P_n(x)$, we shall use the following also (*see* Sansone 1959, pp. 178-80).

$$(2n + 1) P_n(y) = \frac{d}{dy} \{P_{n+1}(y) - P_{n-1}(y)\} \qquad \ldots(1.3)$$

$$P_n(1) = 1, \ P_n(y) = (-1)^n P_n(-y). \qquad \ldots(1.4)$$

Throughout the paper x is a fixed number such that $-1 < x < 1$. We also write, for $0 \leqslant \phi \leqslant \pi$ and $\theta = \cos^{-1} x$

$$t = |\theta - \phi| \qquad \ldots(1.5)$$

$$H_n(\theta, \phi) = G_n(\cos \phi) P_n(\cos \theta) \qquad \ldots(1.6)$$

where

$$G_n(y) = P_{n+1}(y) - P_{n-1}(y). \qquad \ldots(1.7)$$

2. Introduction

Among the absolute summability methods which have been applied to Legendre series (1.1), a few applications of absolute Riesz summability methods to Legendre series are known (Pal and Sharma 1977). The object of this paper is to prove a structural theorem for the absolute Riesz summability of Legendre series which leads to not only some interesting results for the absolute Riesz summability of Legendre series of the different types of Riesz means and order one but also some interesting results for the absolute Riesz summability factors for Legendre series. Precisely we prove the following main theorem:

Theorem — Let T be a positive and non-decreasing function of t such that

$$T \geqslant k/t, \text{ uniformly in } 0 < t < \pi^* \qquad \ldots(2.1)$$

and let the sequence $\{y_n\}$ be non-negative and that

$$\sum_{n=1}^{\infty} n^{-5/4} y_n < \infty \qquad \ldots(2.2)$$

$$\text{(i)} \quad y_n = O(F(n)); \quad \text{(ii)} \int_{k/t}^{T} w^{-1} y(w)\, dw = O\{F(k/t)\} \qquad \ldots(2.3)$$

uniformly in $0 < t < \pi$, where F is a positive and non-decreasing function in $[1, \infty)$. Then

$$\int_0^{\pi} F(k/t) \, | \, df(\cos \phi) \, | < \infty \qquad \ldots(2.4)$$

implies that $\sum_{n=1}^{\infty} a_n y_n P_n(x) \in | R, \lambda(w), 1 |$, whenever $\lambda(w)$ satisfies the following conditions:

*Throughout k is a suitable positive constant, taken for the convenience in the analysis.

$$\left\{\frac{\lambda(n)\,y_n}{n}\right\} \text{ is non-decreasing with } n \geqslant n_0 \qquad ...(2.5)$$

$$\sum_{n \leqslant w} \frac{\lambda(n)\,y_n}{n} = O\left\{\frac{\lambda^2(w)\,y(w)}{w\lambda^{(1)}(w)}\right\} \qquad ...(2.6)$$

$$\int_T^\infty \frac{\lambda^{(1)}(w)\,y(w)}{w\lambda(w)}\,dw = O\{tF(k/t)\} \qquad ...(2.7)$$

uniformly in $0 < t < \pi$.

3. Lemmas

We shall use the following lemmas in the proof of the theorem:

Lemma 1 (Sansone 1959, p. 197) — Let $0 < \theta < \pi$ and $n = 1, 2, 3, \ldots$. Then

$$|\,P_n(\cos\theta)\,| \leqslant 4\sqrt{\frac{2}{n\pi\sin\theta}}.$$

Lemma 2 (Szegö 1967, p. 172) — Let $0 \leqslant \phi \leqslant \pi$. Then

$$|\,G_n(\cos\phi)\,| = O\left(\sqrt{\frac{\sin\phi}{n}}\right).$$

Lemma 3 — Let $n = 1, 2, 3, \ldots$ and c be a fixed positive number. Then for $c/n \leqslant \theta \leqslant \pi - c/n$

$$P_n(\cos\theta) = \sqrt{\frac{2}{n\pi\sin\theta}}\,[\cos\{(n+\tfrac{1}{2})\,\theta - \pi/4\} + O((n\sin\theta)^{-1})].$$

The proof of Lemma 3 follows from (8.21.18) of Szegö (1967).

Lemma 4 — Let c be a fixed positive number and $c/n \leqslant \phi \leqslant \pi - c/n$ and $c/n \leqslant \theta \leqslant \pi - c/n$. Then for $n > 1$

$$H_n(\theta, \phi) = \frac{2}{n\pi}\sqrt{\frac{\sin\phi}{\sin\theta}}\,[\cos\{(n+\tfrac{1}{2})\,(\theta+\phi)\} + \sin\{(n+\tfrac{1}{2})\,(\theta-\phi)\}] + O\left(\frac{n^{-2}}{(\sin\phi)^{3/2}}\right).$$

The proof of Lemma 4 follows from Lemma 3 after some simplifications. Also *see* Pal and Sharma (1977).

Lemma 5 (Obrechkoff 1928 and 1929) — If the series

$$\sum_{n=1}^{\infty} a_n \in | R, \lambda, r | \ (r > 0),$$

then

$$\sum_{n=1}^{\infty} a_n \in | R, \lambda, r' | \ , \text{ where } r' > r.$$

Lemma 6 (Dikshit 1960) — For $k > 0$,

$$| R, e^n, k | \sim | C, 0 | .$$

Lemma 7 (Dikshit 1965) — $| R, \exp \{n/\log n\}, 1 | \subset | N, 1/(n+1) |$.

4. Proof of the Theorem

We have from (1.2)

$$2a_n = (2n+1) \int_{-1}^{1} P_n(y) f(y)\, dy$$

$$= \int_{-1}^{1} f(y) \frac{d}{dy} G_n(y)\, dy$$

[by (1.3) and (1.7)]

$$= - \int_{-1}^{1} G_n(y)\, df(y)$$

$$= \int_{0}^{\pi} G_n(\cos \phi)\, df(\cos \phi)$$

on integrating by parts and using (1.4) and the transformation $y = \cos \phi$. Hence, at $x = \cos \theta$

$$a_n P_n(x) = \tfrac{1}{2} \int_0^{\pi} H_n(\theta, \phi)\, df(\cos \phi). \qquad \text{...(4.1)}$$

Now $\sum_{n=1}^{\infty} a_n P_n(x)\, y_n \in | R, \lambda(w), 1 |$, if

$$I = \int_A^{\infty} \frac{\lambda^{(1)}(w)}{\lambda^2(w)} \left| \sum_{n \leqslant w} \lambda(n)\, y_n a_n P_n(x) \right| dw < \infty.$$

By (4.1),

$$I = \tfrac{1}{2} \int_A^\infty \frac{\lambda^{(1)}(w)}{\lambda^2(w)} \left| \sum_{n \leqslant w} \lambda(n)\, y_n \int_0^\pi H_n(\theta, \phi)\, df(\cos\phi) \right| dw.$$

Splitting up the inner integral in I into

$$\int_0^{c/\sqrt{w}}, \quad \int_{c/\sqrt{w}}^{\pi - c/\sqrt{w}} \quad \text{and} \quad \int_{\pi - c/\sqrt{w}}^{\pi}$$

where c is a positive number, and denoting the respective integrals, thus obtained, by I_1, I_2 and I_3, we have

$$I_1 = \tfrac{1}{2} \int_A^\infty \frac{\lambda^{(1)}(w)}{\lambda^2(w)} \left| \sum_{n \leqslant w} \lambda(n)\, y_n \int_0^{c/\sqrt{w}} H_n(\theta, \phi)\, df(\cos\phi) \right| dw$$

$$= O(1) \int_A^\infty \frac{\lambda^{(1)}(w)}{\lambda^2(w)} \left\{ \sum_{n \leqslant w} \lambda(n)\, y_n \int_0^{c/\sqrt{w}} n^{-1} \sqrt{\sin\phi}\, |\, df(\cos\phi)\, | \right\} dw$$

by using Lemmas 1 and 2. Hence, by using the hypothesis $f(\cos\phi) \in BV[0, \pi]$,

$$I_1 = O(1) \int_A^\infty \frac{\lambda^{(1)}(w)}{\lambda^2(w)} \left| \sum_{n \leqslant w} \frac{\lambda(n)\, y_n}{n^{5/4}} \right| dw < \infty$$

by using (2.2) and Lemma 5. Also using (1.4) and the transformation $\phi = \pi - y$, the convergence of I_3 may be followed on proceeding as in I_1.

Finally, by Lemma 4,

$$I_2 = \frac{1}{\pi} \int_A^\infty \frac{\lambda^{(1)}(w)}{\lambda^2(w)} \left| \sum_{n \leqslant w} n^{-1} \lambda(n)\, y_n \right.$$

$$\times \left. \int_{c/\sqrt{w}}^{\pi - c/\sqrt{w}} \sqrt{\frac{\sin\phi}{\sin\theta}} \cos\{(n + \tfrac{1}{2})(\theta + \phi)\}\, df(\cos\phi) \right| dw$$

$$+ \frac{1}{\pi} \int_A^\infty \frac{\lambda^{(1)}(w)}{\lambda^2(w)} \left| \sum_{n \leqslant w} n^{-1} \lambda(n)\, y_n \right.$$

$$\times \left. \int_{c/\sqrt{w}}^{\pi - c/\sqrt{w}} \sqrt{\frac{\sin\phi}{\sin\theta}} \sin\{(n + \tfrac{1}{2})(\theta - \phi)\}\, df(\cos\phi) \right| dw +$$

(*equation continued on p.* 196)

$$+ O\left\{\int_A^\infty \frac{\lambda^{(1)}(w)}{\lambda^2(w)} \left\{\sum_{n\leqslant w} n^{-2}\lambda(n)\, y_n \int_{c/\sqrt{w}}^{\pi - c/\sqrt{w}} (\sin\phi)^{-3/2} \mid df(\cos\phi) \mid \right\}\right\} dw$$

$$= I_{2\cdot1} + I_{2\cdot2} + O(I_{2\cdot3}), \text{ say.}$$

However, we observe that

$$\sum_{n\leqslant w} n^{-2}\lambda(n)\, y_n \int_{c/\sqrt{w}}^{\pi-c/\sqrt{w}} (\sin\phi)^{-3/2} \mid df(\cos\phi) \mid$$

$$\leqslant \sum_{n\leqslant w} n^{-2}\lambda(n)\, y_n \int_{c/\sqrt{n}}^{\pi} (\sin\phi)^{-3/2} \mid df(\cos\phi) \mid$$

$$= \sum_{n\leqslant w} \lambda(n)\, y_n n^{-5/4}$$

by using $f(\cos\phi) \in BV\,[0, \pi]$. Hence an application of (2.2) and Lemma 5 yield that

$$I_{2\cdot3} < \infty.$$

Further, the sum $\mid \sum_{n\leqslant w} \mid$ in $I_{2\cdot1}$ equals

$$\left| \int_{c/\sqrt{w}}^{\pi-c/\sqrt{w}} \sqrt{\frac{\sin\phi}{\sin\theta}}\, df(\cos\phi) \sum_{n\leqslant w} \frac{\lambda(n)\, y_n}{n} \cos\{(n+\tfrac{1}{2})(\theta+\phi)\} \right|$$

$$= O(1) \int_0^\pi \mid df(\cos\phi) \mid \left| \sum_{n\leqslant w} \frac{\lambda(n)}{n} y_n \cos\{(n+\tfrac{1}{2})(\theta+\phi)\} \right| \qquad \ldots(4.2)$$

and, by (2.5) and Abel's lemma

$$\left| \sum_{n\leqslant w} \frac{\lambda(n)}{n} y_n \cos\{(n+\tfrac{1}{2})(\theta+\phi)\} \right|$$

$$= O(1) + \left| \sum_{n=n_0}^{[w]} \frac{\lambda(n)}{n} y_n \cos\{(n+\tfrac{1}{2})(\theta+\phi)\} \right|$$

$$= O(1) + O\left\{\frac{\lambda(w)\, y(w)}{w} (\theta+\phi)^{-1}\right\}. \qquad \ldots(4.3)$$

Thus (4.2), (4.3) and the hypothesis $f(\cos\phi) \in BV\,[0, \pi]$ yield that

$$I_{2\cdot 1} = O(1) \int_A^\infty \frac{\lambda^{(1)}(w)}{w\lambda(w)}\, y(w)\, dw$$

$$= O(1)$$

by (2.7).

Finally, we consider $I_{2\cdot 2}$, where

$$I_{2\cdot 2} \leqslant \int_0^\pi |\, df(\cos\phi)\,| \int_A^\infty \frac{\lambda^{(1)}(w)}{\lambda^2(w)} \left| \sum_{n \leqslant w} n^{-1}\lambda(n)\, y_n \sin\{(n+\tfrac{1}{2})(\theta-\phi)\} \right| dw$$

$$= \int_0^\theta |\, df(\cos\phi)\,|\, M + \int_\theta^\pi |\, df(\cos\phi)\,|\, M$$

where

$$M = \int_A^\infty \frac{\lambda^{(1)}(w)}{\lambda^2(w)} \left| \sum_{n \leqslant w} n^{-1}\lambda(n)\, y_n \sin(n+\tfrac{1}{2})t \right| dw.$$

Thus to complete the proof of the theorem, we only require to show that

$$M = O(F(k/t)) \qquad \dots (4.4)$$

uniformly in $0 < \phi < \theta$ and $\theta < \phi < \pi$.

The proof of (4.4) in the cases $\phi = 0, \pi$ is trivial. Therefore we consider only (i) $0 < \phi < \theta$, and (ii) $\theta < \phi < \pi$. However the case (ii) $\theta < \phi < \pi$ is similar to that (i) $0 < \phi < \theta$. Thus it is desirable that we should give a proof of (4.4) for $0 < \phi < \theta$ only.

We write for T as defined in (2.1),

$$M = \left\{ \int_A^{k/t} + \int_{k/t}^{T} + \int_T^\infty \right\} \frac{\lambda^{(1)}(w)}{\lambda^2(w)} \left| \sum_{n \leqslant w} n^{-1}\lambda(n)\, y_n \sin(n+\tfrac{1}{2})t \right| dw$$

$$= M_1 + M_2 + M_3, \text{ say.}$$

However,

$$M_1 = O\left\{ t \int_A^{k/t} w\, \frac{\lambda^{(1)}(w)}{\lambda^2(w)} \left| \sum_{n \leqslant w} \frac{\lambda(n)}{n}\, y_n \right| dw \right\}$$

(equation continued on p. 198)

$$= O\left\{t \int_{A}^{k/t} y(w)\, dw\right\} \text{ (by (2.6))}$$

$$= O(F(k/t))$$

using (2.3i). And, since $|\sin nt| \leqslant 1$, we get

$$M_2 = O\left\{\int_{k/t}^{T} \frac{\lambda^{(1)}(w)}{\lambda^2(w)} \left(\sum_{n \leqslant w} n^{-1}\lambda(n)\, y_n\right) dw\right\}$$

$$= O\left\{\int_{k/t}^{T} \frac{y(w)}{w}\, dw\right\} \text{ (by (2.6))}$$

$$= O(F(k/t))$$

using (2.3ii). Finally, by using (2.5) and Abel's lemma we observe that

$$\sum_{n \leqslant w} \frac{y_n \lambda(n) \sin nt}{n} = O\left(\frac{\lambda(w)\, y(w)}{wt}\right)$$

uniformly in $0 < t < \theta$. Hence

$$M_3 = O\left\{\frac{1}{t} \int_{T}^{\infty} \frac{\lambda^{(1)}(w)}{w\lambda(w)}\, dw\right\}$$

$$= O(F(k/t))$$

using (2.7), uniformly in $0 < \phi < \theta$.

This completes the proof of the theorem.

5. Corollaries

In this section we obtain a number of interesting results, in the form of corollaries from the main theorem.

Corollary 1 — Let

$$\int_0^{\pi} \log\left(\frac{2\pi}{t}\right) |\, df(\cos\phi)\, | < \infty.$$

Then

$$\sum_{n=1}^{\infty} a_n P_n(x) \in |\, R, \exp(w^{\alpha}), 1\, |, \ (0 < \alpha < 1).$$

PROOF : Let in the main theorem,

$$F(k/t) = \log\left(\frac{2\pi}{t}\right),\ \lambda(w) = \exp(w^\alpha)\ (0 < \alpha < 1)$$

$$T = (2\pi/t)^{1/(1-\alpha)} \text{ and } y_n = 1 \text{ for all } n.$$

Then we observe that (2.2) and (2.3) are satisfied, since

$$\int_{k/t}^{T} 1/w\, dw = O(\log(2\pi/t)), \text{ uniformly in } 0 < t < \pi.$$

Further, $\left\{\frac{\exp(n^\alpha)}{n}\right\}$ is monotonic increasing with $n \geqslant n_0$, where $n_0 = 1 + [1/\alpha]^{1/\alpha}$.

And

$$\sum_{n \leqslant w} \frac{\lambda(n)}{n} = O(1) + \int_{n_0}^{w} \frac{\exp(x^\alpha)}{x}\, dx$$

$$= O(1) + O\left(\frac{\exp(w^\alpha)}{w^\alpha}\right)$$

$$= O\left(\frac{\exp(w^\alpha)}{w^\alpha}\right) = O\left(\frac{w^{-1}\lambda^2(w)}{\lambda^{(1)}(w)}\right).$$

Hence (2.5) and (2.6) also hold.

Again, uniformly in $0 < t < \pi$,

$$\int_{T}^{\infty} \frac{w^{\alpha-1}}{w}\, dw = O(t)$$

$$= O(t \log(2\pi/t)).$$

Hence Corollary 1 follows from the main theorem of the paper.

Corollary 2 — Let, for $k > \pi e^2$,

$$\int_{0}^{\pi} \log\log(k/t)\ |\ df(\cos\phi)\ |\ < \infty.$$

Then

$$\sum_{n=1}^{\infty} a_n P_n(x) \in |\ R, \exp\{(\log w)^\Delta\}, 1\ |\ , (\Delta > 1;\ \text{however large}).$$

PROOF : On taking $y_n = 1$ for all n, $F(k/t) = \log\log\left(\frac{\pi e^2}{t}\right)$,

$$T = \frac{\pi e^2}{t}\left(\log\left(\frac{\pi e^2}{t}\right)\right)^{\Delta-1} \text{ and } \lambda(w) = \exp\{(\log w)^\Delta\}\ (\triangle > 1)$$

in the main theorem and observing that (*see* (3.4) of Chandra, 1970)

$$\sum_{n \leqslant w} \frac{\lambda(n)}{n} = O\{\lambda(w)\,(\log w)^{1-\Delta}\}$$

the proof of the corollary may be followed.

Corollary 3 — Let, for $b > 0$,

$$\int_0^\pi t^{-b/(1+b)}\,|\,df(\cos\phi)\,| < \infty.$$

Then

$$\sum_{n=1}^{\infty} a_n P_n(x) \in |\,R, \exp\{w/(\log w)^{(1+b)/b}\}, 1\,|\,.$$

PROOF : The proof of Corollary 3 follows from the main theorem on taking $y_n = 1$, $F(k/t) = t^{-b/(1+b)}$ $(b > 0)$, $T = \exp\{t^{-b/(1+b)}\}$ and $\lambda(w) = \exp\{w/(\log w)^{(b+1)/b}\}$, and observing the fact that

$$\sum_{n \leqslant w} \frac{\lambda(n)}{n} = O\left\{\frac{\lambda(w)}{w}\,(\log w)^{(b+1)/b}\right\}.$$

Corollary 4 — Let $f(\cos\phi) \in BV\,[0, \pi]$. Then

$$\sum_{n=1}^{\infty} \frac{a_n P_n(x)}{(\log n)^{1+\epsilon}} \in |\,C, 0\,|\,, \text{ where } \epsilon > 0.$$

PROOF : Let $y_n = 1/(\log n)^{1+\epsilon}$ $(\epsilon > 0)$, $F(k/t) = 1$ for all t, $T = \exp(t^{-1/\epsilon})$ and $\lambda(n) = e^n$ in the main theorem. Then we observe that (2.2) and (2.3) hold.

Also it is easy to see that $\left\{\frac{\lambda(n)}{n(\log n)^{1+\epsilon}}\right\}$ is monotonic increasing for some suitable $n \geqslant n_0$,

$$\sum_{n \leqslant w} \frac{\lambda(n)}{n(\log n)^{1+\epsilon}} = O\left(\frac{\lambda(w)}{w(\log w)^{1+\epsilon}}\right)$$

and

$$\int_T^\infty 1/w(\log w)^{1+\epsilon}\,dw = O(t).$$

Hence (2.5), (2.6) and (2.7) are satisfied. Consequently the main theorem yields that

$$\sum_{n=1}^{\infty} \frac{a_n P_n(x)}{(\log n)^{1+\varepsilon}} \in | R, e^n, 1 | .$$

However, by using Lemma 6, the proof of the corollary follows.

Corollary 5 — Let

$$\int_0^{\pi} \log (2\pi/t) \mid df(\cos \phi) \mid < \infty.$$

Then

$$\sum_{n=1}^{\infty} \frac{a_n P_n(x)}{\log (n+1)} \in | R, \exp \{w/(\log w)\}, 1 | .$$

PROOF : Taking $F(k/t) = \log (2\pi/t)$, $y_n = 1/\log (n+1)$, $T = \exp (2\pi/t)$ and $\lambda(w) = \exp \{w/(\log w)\}$ $(w > 1)$ in the main theorem, the proof of the corollary follows.

Remark : It is known (Pandey 1967) that

$$\sum_{n=1}^{\infty} \frac{a_n P_n(x)}{\log (n+1)} \in | N, 1/(n+1) |$$

whenever $f(\cos \phi) \in BV\,[0, \pi]$. But, by Lemma 7, it follows that the summability method $| R, \exp \{w/\log w\}, 1 |$ is contained in $| N, 1/(n+1) |$ and hence the stronger condition in the corollary is justified.

Corollary 6 — Let for $\alpha > 0$, $\frac{1}{4} > \beta > 0$, $1 > \alpha + \beta$ and $\delta = \beta/(1-\alpha)$

$$\int_0^{\pi} t^{-\delta} \mid df(\cos \phi) \mid < \infty.$$

Then

$$\sum_{n=1}^{\infty} a_n P_n(x)\, n^{\beta} \in | R, \exp (n^{\alpha}), 1 | .$$

PROOF : On taking $F(k/t) = t^{-\delta}$, $y_n = n^{\beta}$, $T = (k/t)^{\delta/\beta}$ and $\lambda(w) = \exp (w^{\alpha})$, whenever the real numbers α, β, δ satisfy the conditions of the corollary, the proof of Corollary 6 follows from the main theorem.

Corollary 7 — Let, for $0 < b < 1$, $c > 0$, and $d > 0$, $bc = 1 + d$ and let

$$\int_0^\pi t^{-b} \mid df(\cos \phi) \mid < \infty.$$

Then

$$\sum_{n=1}^{\infty} a_n P_n(x) (\log (n + 1))^d \in \mid R, \exp \{w/(\log w)^c\}, 1 \mid .$$

PROOF : On taking $F(k/t) = t^{-b}$, $\lambda(w) = \exp \{w/(\log w)^c\}$, $y_n = (\log (n + 1))^d$ and $T = 3 \exp (t^{-b/(1+d)})$ in the main theorem, the proof is immediate.

Corollary 8 — Let $\delta > 0$ and $0 < \alpha < 1$ and let

$$\int_0^\pi (\log k/t)^{1+\delta} \mid df(\cos \phi) \mid < \infty.$$

Then

$$\sum_{n=1}^{\infty} a_n P_n(x) (\log (n + 1))^\delta \in \mid R, \exp (w^\alpha), 1 \mid .$$

PROOF : Taking $F(k/t) = (\log k/t)^{1+\delta}$, $y_n = (\log (n + 1))^\delta$, $\lambda(w) = \exp (w^\alpha)$ and $T = (k/t)^{1/(1-\alpha)}$ the proof of the corollary can be completed from the main theorem.

Corollary 9 — Let, for $c > 0$, $0 < d < 1$ and $\Delta \geqslant 1$, $1 + c > d(\Delta - 1)$ and let

$$\int_0^\pi t^{-d} \mid df(\cos \phi) \mid < \infty.$$

Then

$$\sum_{n=1}^{\infty} \frac{a_n P_n(x)\, n^d}{(\log (n + 1))^{1+c}} \in \mid R, \exp \{(\log w)^\Delta\}, 1 \mid .$$

PROOF : Taking $F(k/t) = t^{-d}$, $\lambda(w) = \exp \{(\log w)^\Delta\}$, $y_n = n^d/(\log (n + 1))^{1+c}$ and $T = (2\pi/t) (\log 2\pi/t)^{1+c}$, the proof of the corollary follows from the main theorem.

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