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GENERAL FINITE DIFFERENCE SCHEMES FOR THE HEAT EQUATIONS

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General implicit finite difference schemes of LOD, ADI, and SOD types are derived taking into account the heat equation in two space variables and using cubic spline technique. Stability of the schemes is examined. For particular values of the parameter, these schemes turn out to be of high order and are found to be unconditionally stable. The formulae due to Mitchell and Fairweather, Peaceman and Rachford, Hubbard, Samarskii and Andreev and a general explicit scheme are obtained as special cases of the general implicit schemes. Relative merits of these formulae are discussed and illustrated by using numerical test examples. Finite difference schemes are also obtained for the heat equation in N-dimensional heat equation.

1. INTRODUCTION

The heat equation is of common occurrence in formulating a mathematical model of a physical problem arising in science and engineering. It is known that one has often to resort to numerical methods for solving such problems. There are several numerical schemes available in the literature for such problems. But a systematic discussion for choosing the 'best' scheme for solving a heat equation problem is not available. In the present paper, an attempt has been made to derive general implicit finite difference schemes of ADI, LDO and SOD types for the heat equation. Stability and accuracy of these schemes have been discussed and it is found that some high order schemes are unconditionally stable. The schemes due to Mitchell and Fairweather (1969), Peaceman and Rachford (1955), Samarskii and Andreev (1963), Hubbard (1965) and the general explicit scheme (Mitchell 1969) have been obtained as special cases of the general formulae. Results have been further generalized to the heat equation in N-space variables.

2. FORMULATION OF THE PROBLEM

Consider the heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \text{ on } R = [(x, y) : 0 \leqslant x, y \leqslant 1] \, X \, [0 \leqslant t \leqslant T] \quad \ldots(2.1)$$

with the initial condition

$$u(x, y, 0) = g(x, y); \; 0 \leqslant x, y \leqslant 1 \quad \ldots(2.2)$$

and the boundary condition

$$u(x, y, t) = f(x, y, t) \qquad \text{...(2.3)}$$

$(x, y, t) \in \partial RX\ (0 \leqslant t \leqslant T)$, where ∂R is the boundary of R, $u(x, y, t)$ is the unknown function and g, f are known functions; we assume these functions to have sufficient smoothness for maintaining the order of accuracy of the difference schemes to be considered. We denote the discrete approximation of $u(x, y, t)$ at the point (ih, jh, nk) by $U^n_{i,j}$, $(i, j = 0, 1, 2, \ldots, P;\ n = 0, 1, 2, \ldots)$, where h is the mesh step both in x- and y-directions and k represents the increment in time.

3. ADI Schemes

We write eqn. (2.1) as

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial y^2}. \qquad \text{...(3.1)}$$

We approximate the time derivative by the forward differences and the space derivative with respect to y by the central differences in eqn. (3.1), while a cubic spline approximation is used for the term on the left-hand side of (3.1). Thus,

$$\theta M^{(n+1)/2}_{i,j} + (1 - \theta) M^n_{i,j} = \frac{2}{k}\left(U^{(n+1)/2}_{i,j} - U^n_{i,j}\right) - \frac{1}{h^2}\delta^2_y U^n_{i,j} \qquad \text{...(3.2)}$$

where

$$\delta^2_y U^n_{i,j} = U^n_{i,j-1} - 2U^n_{i,j} + U^n_{i,j+1}$$

$0 \leqslant \theta \leqslant 1$, $M^n_{i,j} = S''_{j,n}(x_i)$; $S_{j,n}(x)$ denoting the cubic spline interpolating the values $U^n_{i,j}$ on the nth time level and the jth y-line. From this equation, an implicit finite difference formula in the x-direction is derived (*cf.* Papamichael and Whiteman 1973).

$$(1 + a\delta^2_x)\left(U^{(n+1)/2}_{i,j} - U^n_{i,j}\right) = \frac{r}{2}\left(\delta^2_x + \delta^2_y + \tfrac{1}{6}\delta^2_x\,\delta^2_y\right) U^n_{i,j} \qquad \text{...(3.3)}$$

where

$$a = \tfrac{1}{6} - \frac{r\theta}{2},\ r = \frac{k}{h^2}.$$

Hereafter, we shall denote $U^n_{i,j}$ by U^n except in some equations containing other terms also.

Similarly one gets an implicit finite difference formula in the y-direction as

$$(1 + a\delta_y^2)(U^{n+1} - U^{(n+1)/2}) = \frac{r}{2}(\delta_x^2 + \delta_y^2 + \tfrac{1}{6}\delta_x^2\,\delta_y^2)\,U^{(n+1)/2}. \quad ...(3.4)$$

Equations (3.3) and (3.4) give the ADI scheme for solving the given problem. On eliminating the intermediate value $U^{(n+1)/2}$ from eqns. (3.3) and (3.4), one gets

$$(1 + a\delta_x^2)(1 + a\delta_y^2)(U^{n+1} - U^n) = r(\delta_x^2 + \delta_y^2 + b\,\delta_x^2\,\delta_y^2$$

$$+ c(\delta_x^4 + \delta_y^4) + d(\delta_x^2\,\delta_y^4 + \delta_x^4\,\delta_y^2) + e\,\delta_x^4\,\delta_y^4)\,U^n \quad ...(3.5)$$

where $b = a + \tfrac{1}{6} + \dfrac{r}{2}$, $c = \dfrac{a}{2} + \dfrac{r}{4}$, $d = \dfrac{(a + r)}{12}$ and $e = \dfrac{r}{144}$.

Using von Neumann method of stability, scheme (3.5) is unconditionally stable for $\tfrac{2}{3} \leqslant \theta \leqslant 1$ and conditionally stable for $0 \leqslant \theta \leqslant \tfrac{2}{3}$ with the condition

$$r \leqslant \frac{1}{[2(2 - 3\theta)]}.$$

The principal part of the truncation error of (3.5) is

$$\frac{k^2}{4}(1 - \theta)\frac{\partial^2 u}{\partial t^2}.$$

By taking $\theta = \tfrac{2}{3}$ in eqn. (3.5), one gets optimal unconditionally stable scheme of the family. The intermediate boundary value to be used in (3.3) can be taken as

$$U^{(n+1)/2} = f^{(n+1)/2}. \quad ...(3.6)$$

By taking $\theta = 1$ in (3.5), one gets a scheme of high order accuracy $O(k^2 + h^4)$ which is unconditionally stable. But the accuracy of the scheme at the nodal point adjacent to the boundary does not remain $O(k^2 + h^4)$ as at the interior points; this problem arises because of the presence of the fourth order differences in the scheme. It affects the global accuracy of the scheme.

The above results can also be generalized for the equation in N-space variables. In general, we consider

$$\frac{\partial u}{\partial t} = \sum_{i=1}^{N} \frac{\partial^2 u}{\partial x_i^2} \quad ...(3.7)$$

with appropriate initial and boundary conditions. For this equation finite difference implicit scheme is

$$\prod_{i=1}^{N} (1 + a_2\delta^2_{x_i})\, U^{n+1} = \prod_{i=1}^{N} \Big\{(1 + a_2\delta^2_{x_i}) + \frac{r}{N}\Big[\sum_{j=1}^{N} \delta^2_{x_j} + \tfrac{1}{6}\delta^2_{x_i} \sum_{\substack{m=1\\ m\neq i}}^{N} \delta^2_{x_m}\Big]\Big\} U^n \qquad \ldots(3.8)$$

where $a_2 = \frac{1}{6} - \frac{r\theta}{N}$.

The condition of stability for the scheme (3.8) is

$$r \leqslant \frac{N}{[2(2 + N - 6\theta)]}.$$

Thus, for $N = 4$, it is unconditionally stable for $\theta = 1$ and optimum. But for $N \geqslant 5$, it is always conditionally stable. The order of accuracy of the scheme (3.8) is the same as that for the two space variables, that is $O(k + h^2)$.

4. Split Operator Difference Scheme (SOD)

Using the cubic spline approximation and central difference, eqn. (3.1) gives

$$\theta M^{n+1}_{i,j} - \theta M^{n}_{i,j} + M^{n+1*}_{i,j} = \frac{1}{k}(U^{n+1}_{i,j} - U^{n}_{i,j}) = \frac{1}{h^2}\delta^2_y U^{n+1*}_{i,j} \qquad \ldots(4.1)$$

for which the finite difference form is

$$(1 + \mu\delta^2_x)(U^{n+1} - U^n) = r(\delta^2_x + \delta^2_y + \tfrac{1}{6}\delta^2_x \delta^2_y)\, U^{n+1*} \qquad \ldots(4.2)$$

where $\mu = \frac{1}{6} - r\theta$.

Here $U^{n+1*}_{i,j}$ denotes the intermediate value which can be computed as

$$(1 + \mu\delta^2_y)\, U^{n+1*}_{i,j} = U^{n}_{i,j}\,. \qquad \ldots(4.3)$$

For a rectangular region, one can eliminate U^{n+1*} from (4.2) and (4.3) to yield

$$(1 + \mu\delta^2_x)(1 + \mu\delta^2_y)(U^{n+1} - U^n) = r(\delta^2_x + \delta^2_y + \tfrac{1}{6}\delta^2_x\delta^2_y)\, U^n. \qquad \ldots(4.4)$$

Using von Neumann method of stability, one finds that the scheme (4.4) is unconditionally stable for $\theta \geqslant \frac{1}{2}$ and conditionally stable for $0 < \theta < \frac{1}{2}$ with the condition

$$r \leqslant (1 - \theta - \sqrt{(1 - 2\theta)})/12\theta^2. \qquad \ldots(4.5)$$

The principal part of the truncation error of this scheme is

$$\left(\frac{kh^2}{12}\right)(1 + 6r - 12r\theta)\frac{\partial^2 u}{\partial t^2}.$$

The scheme (4.4) is a general scheme having the parameter θ. By taking special values of θ, one can derive several schemes some of which are well known in the literature (Mitchell and Fairweather 1969). In the following we give a summary of these special cases of (4.4):

θ	*Scheme*	*Accuracy*	*Stability*
$\frac{1}{2} + \frac{1}{12r}$	Mitchell-Fairweather scheme (1969)	$O(k^2 + h^4)$	unconditional
$1 + \frac{1}{6r}$	Smarskii-Andreev scheme (1963)	$O(k + h^2)$	unconditional
$\frac{1}{2}$	"Optimum" scheme of family (4.4)	$O(k^2 + h^2)$	unconditionally stable

"Optimum scheme" of the family (4.4) can be written as

$$\left(1 + \left(\tfrac{1}{6} - \frac{r}{2}\right)\delta_x^2\right)\left(1 + \left(\tfrac{1}{6} - \frac{r}{2}\right)\delta_y^2\right)(U^{n+1} - U^n)$$

$$= r(\delta_x^2 + \delta_y^2 + \tfrac{1}{6}\delta_x^2\,\delta_y^2)\,U^n.$$

For computations, some of the splittings can be taken as follows:

$$\left.\begin{aligned}(1 + \mu\delta_x^2)\,U^{n+1*} &= r(\delta_x^2 + \delta_y^2 + \tfrac{1}{6}\delta_x^2\,\delta_y^2 + (1 + \mu\delta_x^2)\\ &\qquad \times (1 + \mu\delta_y^2))\,U^n\\ (1 + \mu\delta_y^2)\,U^{n+1} &= U^{n+1*}.\end{aligned}\right\} \quad \ldots(4.6)$$

$$\left.\begin{aligned}(1 + \mu\delta_x^2)\,U^{n+1*} &= r(\delta_x^2 + \delta_y^2 + \tfrac{1}{6}\delta_x^2\,\delta_y^2)\,U^n\\ (1 + \mu\delta_y^2)(U^{n+1} - U^n) &= U^{n+1*}.\end{aligned}\right\} \quad \ldots(4.7)$$

$$\left.\begin{aligned}(1 + \mu\delta_x^2)\,U^{n+1*} &= r(\delta_x^2 + \delta_y^2 - \frac{1}{6\mu}\delta_y^2)\,U^n\\ (1 + \mu\delta_y^2)(U^{n+1} - U^n) &= U^{n+1*} + \frac{r}{6\mu}\delta_y^2\,U^n.\end{aligned}\right\} \quad \ldots(4.8)$$

$$\left.\begin{aligned}(1 + \mu\delta_x^2)\, U^{n+1*} &= \frac{r}{\mu}[-1 + (\mu - \tfrac{1}{6})\,\delta_y^2]\, U^n \\ (1 + \mu\delta_y^2)\,(U^{n+1} - U^n) &= U^{n+1*} + \frac{r}{\mu}(1 + \tfrac{1}{6}\delta_y^2)\, U^n.\end{aligned}\right\} \quad \ldots(4.9)$$

The most efficient splitting from the computational point of view is given by (4.9), as it requires the computation of $\delta_y^2 U^n$ only. In all these split operator schemes, the intermediate boundary condition must be calculated explicitly from the second formula for maintaining the global order of accuracy.

In general, in N-space dimensions, finite difference approximation for (3.7) is

$$\prod_{i=1}^{N} (1 + \mu\delta_{x_i}^2)\,(U^{n+1} - U^n) = r[\sum_{i=1}^{N} \delta_{x_i}^2 + \tfrac{1}{6}\delta_{x_1}^2 \sum_{j=2}^{N} \delta_{x_j}^2]\, U^n. \quad \ldots(4.10)$$

In this case, it is difficult to find precisely the stability condition. But by taking $N = 4$, it is unconditionally stable for $1 \geqslant \theta \geqslant \frac{1}{2}$ and when $N \geqslant 5$, it is conditionally stable. The principal part of the truncation error is of $O(k^2 + kh^2)$. It seems that higher order accuracy schemes cannot be deduced from the general case when $N \geqslant 3$.

New type of SOD scheme — If we neglect the higher order terms $(\delta_x^2\,\delta_y^4 + \delta_x^4\,\delta_y^2)$ and $\delta_x^4\,\delta_y^4$ in (3.5), the order of accuracy of the resulting scheme remains unaltered. Hence, we get

$$(1 + a\delta_x^2)(1 + a\delta_y^2)(U^{n+1} - U^n)$$

$$= r[\delta_x^2 + \delta_y^2 + \tfrac{1}{6}\delta_x^2\,\delta_y^2 + c(\delta_x^2 + \delta_y^2)^2]\, U^n. \quad \ldots(4.11)$$

For computational purposes, we can split eqn. (4.11) in the form

$$(1 + a\delta_x^2)\, U^{n+1*} = r(\delta_x^2 + \delta_y^2 + \tfrac{1}{6}\delta_x^2\,\delta_y^2)\, U^n$$

$$U^{n+1**} = rc(\delta_x^2 + \delta_y^2)\, U^n$$

$$(1 + a\delta_x^2)\, U^{n+1***} = (\delta_x^2 + \delta_y^2)\, U^{n+1**}$$

$$(1 + a\delta_y^2)\,(U^{n+1} - U^n) = U^{n+1*} + U^{n+1***}.$$

The intermediate values U^{n+1*}, U^{n+1**} and U^{n+1***} can be calculated from

$$U^{n+1*} = (1 + a\delta_y^2)(f^{n+1} - f^n)$$

$$U^{n+1**} = rc(f^{n+1} - f^n)$$

$$U^{n+1***} = (1 - r + a\delta_y^2)(f^{n+1} - f^n).$$

By using von Neumann method for stability, this scheme is found to be unconditionally stable for $\theta = 1$ with an order of accuracy $O(k^2 + h^4)$, it is unstable for all other values of the parameter θ. The unconditionally stable scheme ($\theta = 1$) is

$$\left(1 + \left(\frac{1}{6} - \frac{r}{2}\right)\delta_x^2\right)\left(1 + \left(\frac{1}{6} - \frac{r}{2}\right)\delta_y^2\right)(U^{n+1} - U^n)$$

$$= r\left(\delta_x^2 + \delta_y^2 + \tfrac{1}{6}\delta_x^2\,\delta_y^2 + \frac{1}{12}(\delta_x^2 + \delta_y^2)^2\right)U^n. \quad \text{...(4.12)}$$

The global accuracy of the scheme gets affected due to the approximations used at nodal points adjacent to the boundary.

5. LOD Schemes

We write eqn. (2.1) as

$$\tfrac{1}{2}\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \quad \text{...(5.1)}$$

$$\tfrac{1}{2}\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial y^2}. \quad \text{...(5.2)}$$

By making use of the cubic spline technique one can approximate (5.1) as:

$$\theta M_{i,j}^{(n+1)/2} + (1 - \theta)\, M_{i,j}^{n} = \frac{1}{k}(U_{i,j}^{(n+1)/2} - U_{i,j}^{n}).$$

On simplification, we get

$$(1 + \mu\,\delta_x^2)\,(U^{(n+1)/2} - U^n) = r\,\delta_x^2\,U^n. \quad \text{...(5.3)}$$

One can derive a similar expression at a higher time level by making use of eqn. (5.2) in the form

$$(1 + \mu\,\delta_y^2)\,(U^{n+1} - U^{(n+1)/2}) = r\,\delta_y^2\,U^{(n+1)/2}. \quad \text{...(5.4)}$$

Now, on eliminating the intermediate value $U^{(n+1)/2}$, we get a generalized LOD scheme in the form

$$(1 + \mu\,\delta_x^2)(1 + \mu\,\delta_y^2)\,U^{n+1} = (1 + \beta\,\delta_x^2)(1 + \beta\,\delta_y^2)\,U^n \quad \text{...(5.5)}$$

where $\beta = \mu + r$.

On examining (5.5), we find the condition for LOD scheme as

$$\beta = \mu + r. \quad \text{...(5.6)}$$

For computational purposes (cf. Examples 1 and 2) scheme (5.5) can be written in the ADI form as

$$\left.\begin{aligned}(1+\mu\delta_x^2)\,U^{n+1*} &= (1+\beta\delta_y^2)\,U^n\\ (1+\mu\delta_y^2)\,U^{n+1} &= (1+\beta\delta_x^2)\,U^{n+1*}.\end{aligned}\right\} \quad \ldots(5.7)$$

It may be remarked that the well-known schemes of LOD type due to Mitchell and Fairweather (1969), Peaceman and Rachford (1955), Hubbard (1965) and general explicit scheme satisfy this condition (5.6).

By using von Neumann method for stability, the scheme (5.5) is found to be unconditionally stable for $\theta \geqslant \frac{1}{2}$ and conditionally stable for $0 \leqslant \theta < \frac{1}{2}$ with the condition

$$r \leqslant \frac{1}{[6(1-2\theta)]}.$$

The principal part of the truncation error of the scheme (5.5) is

$$k^2\left[\left(\tfrac{1}{2}+\frac{1}{12r}-\theta\right)\frac{\partial^2 u}{\partial t^2}-2\left(\tfrac{1}{2}+\frac{1}{12r}-\theta\right)\frac{\partial^4 u}{\partial x^2\partial y^2}\right].$$

Some of the well-known formulae of LOD type can be derived from (5.5) as follows:

θ	*Scheme*	*Accuracy*	*Stability*
$\frac{1}{2}+\frac{1}{12r}$	Mitchell-Fairweather scheme (1969)	$O(k^2+h^4)$	unconditional
$\frac{1}{2}+\frac{1}{6r}$	Peaceman-Rachford scheme (1955)	$O(k^2+h^2)$	unconditional
$1+\frac{1}{6r}$	Hubbard scheme (1965)	$O(k+h^2)$	unconditional
$\frac{1}{6r}$	General explicit scheme (Mitchell 1969)	$O(k+h^2)$	conditionally stable
$\frac{1}{2}$	"Optimum" scheme of the family (5.5)	$O(k^2+h^2)$	unconditionally stable

The "optimum" unconditionally stable scheme of the family (5.5) is

$$\left(1+\left(\tfrac{1}{6}-\frac{r}{2}\right)\delta_x^2\right)\left(1+\left(\tfrac{1}{6}-\frac{r}{2}\right)\delta_y^2\right)U^{n+1}$$
$$=\left(1+\left(\tfrac{1}{6}+\frac{r}{2}\right)\delta_x^2\right)\left(1+\left(\tfrac{1}{6}+\frac{r}{2}\right)\delta_y^2\right)U^{n}.$$

For computational purposes, one can also write these schemes in the ADI forms. This is possible since the commutation of the operators is valid in these formulae. The relation between ADI and LOD splitting is discussed in Gourlay and Mitchell (1969) and Morris and Gourlay (1973). The expressions for the intermediate boundary values for both LOD and ADI types of splittings are also given by Gourlay and Mitchell (1972).

In general, in N-space dimensions, finite difference approximation for (3.7) is

$$\prod_{i=1}^{N} (1 + \mu\delta_{x_i}^2)\, U^{n+1} = \prod_{i=1}^{N} (1 + \beta\delta_{x_i}^2)\, U^n. \qquad \ldots(5.8)$$

Equation (5.8) satisfies the condition (5.6) for LOD schemes. The stability condition for the scheme is the same as for the schemes discussed earlier. The principal part of the truncation error of the scheme (5.8) is of the order $O(k^2 + kh^2)$. For $\theta = \frac{1}{2} + 1(12r)$ in eqn. (5.8), we get a higher accuracy scheme due to Fairweather *et al.* (1967). For $\theta = \frac{1}{2}$ in eqn. (5.8), we get optimum scheme with an order of accuracy $O(k^2 + h^2)$.

6. Numerical Examples

Example 1 — Consider eqn. (2.1) with the initial condition

$$u(x, y, 0) = \sin \pi x \sin \pi y;\ 0 \leqslant x, y \leqslant 1$$

and boundary condition $u(x, y, t) = 0$ on the boundary ∂R. Its exact solution is

$$u(x, y, t) = \sin \pi x \sin \pi y \exp(-2\pi^2 t).$$

Example 2 — Consider eqn. (2.1) with the initial condition

$$u(x, y, 0) = (\sin \pi x + \cos \pi x)(\sin \pi y + \cos \pi y),\ 0 \leqslant x, y \leqslant 1,$$

and the boundary conditions

$$u(0, y, t) = (\sin \pi y + \cos \pi y) \exp(-2\pi^2 t),\ 0 \leqslant y \leqslant 1,\ t \geqslant 0$$

$$u(1, y, t) = -(\sin \pi y + \cos \pi y) \exp(-2\pi^2 t);\ 0 \leqslant y \leqslant 1,\ t \geqslant 0$$

$$u(x, 0, t) = (\sin \pi x + \cos \pi x) \exp(-2\pi^2 t);\ 0 \leqslant x \leqslant 1,\ t \geqslant 0$$

$$u(x, 1, t) = -(\sin \pi x + \cos \pi x) \exp(-2\pi^2 t);\ 0 \leqslant x \leqslant 1,\ t \geqslant 0.$$

Its exact solution is

$$u(x, y, t) = (\sin \pi x + \cos \pi x)(\sin \pi y + \cos \pi y) \exp(-2\pi^2 t).$$

Tables I and II include a summary of the solutions of these problems at some typical mesh points; the behaviour of the solutions is similar at the other mesh points of the domain. In Tables I and II, we have included the results for some numerical schemes which are unconditionally stable and are of high orders. It is shown that Mitchell and Fairweather scheme gives the most accurate numerical results for Example 2 and the scheme (4.11) for Example 1.

7. Discussion

A general finite difference approximation to the heat equation with parameter θ seems to be very useful in deriving higher order accuracy schemes. For particular values of the parameter θ, we have derived several schemes which are well known in the literature. But for some other particular values of the parameter θ, we are able to derive other useful schemes. For example, in the two dimensional case, we are able to get 'optimum' unconditionally stable schemes of ADI, LOD and SOD types with an order of accuracy $O(k + h^2)$ as well as the schemes of the high order $O(k^2 + h^4)$. It turns out that the high order schemes of SOD and LOD types are the well-known schemes known as Mitchell and Fairweather scheme. In the case of higher space variables, the method does not provide high accuracy ADI type of schemes of $O(k^2 + h^4)$ and schemes of SOD type of $O(k^2 + h^4)$ for a particular value of θ. But we are able to obtain high accuracy LOD type of scheme due to Fairweather and Mitchell of a particular value for the parameter θ.

TABLE I

Numerical solution at the point (0.35, 0.35) *by several numerical schemes* $h = 0.05$, $r = 1.5$ *and* $t = 5\,nk$

n	Exact solution	ADI scheme eqn. (3.6) $\theta = 1$	New SOD scheme eqn. (4.11) $\theta = 1$	SOD scheme eqn. (4.4) $\theta = \frac{1}{2} + (1/12r)$	LOD scheme in ADI splitting eqn. (5.7) $\theta = \frac{1}{2} + (1/12r)$
1	0.452042	0.452023	0.452037	0.452017	0.452023
2	0.312206	0.312181	0.312199	0.312173	0.312181
3	0.215628	0.215600	0.215621	0.215594	0.215601
4	0.148925	0.148898	0.148919	0.148895	0.148900
5	0.102857	0.102832	0.102851	0.102831	0.102835
6	0.071039	0.071018	0.071034	0.071018	0.071021
7	0.049063	0.049047	0.049060	0.049047	0.049049
8	0.033886	0.033873	0.033883	0.033873	0.033874
9	0.023404	0.023394	0.023401	0.023393	0.023395
10	0.016165	0.016157	0.016162	0.016156	0.016157

TABLE II

Numerical solution at the point (0.45, 0.35) *by several numerical schemes* $h = 0.05$, $r = 1.5$ *and* $t = 5nk$

n	Exact solution	ADI scheme eqn. (3.6) $\theta = 1$	New SOD scheme eqn. (4.11) $\theta = 1$	SOD scheme eqn. (4.4) $\theta = \frac{1}{2} + (1/12r)$	LOD scheme in ADI splitting eqn. (5.7) $\theta = \frac{1}{2} + (1/12r)$
1	1.215609	1.214710	1.214112	1.215561	1.215561
2	0.839571	0.837762	0.837276	0.839511	0.839511
3	0.579856	0.578134	0.577784	0.579800	0.579800
4	0.400483	0.399116	0.388866	0.400435	0.400435
5	0.276597	0.275585	0.275408	0.276558	0.276558
6	0.191034	0.190307	0.190184	0.191003	0.191003
7	0.131939	0.131426	0.131341	0.131916	0.131916
8	0.091125	0.090766	0.090706	0.091107	0.091107
9	0.062938	0.062686	0.062444	0.062923	0.062923
10	0.043470	0.043293	0.043267	0.043457	0.043457

For ADI and SOD types of schemes, the stability region becomes smaller as one goes to the higher dimensional case and for $N \geqslant 5$ the schemes do not remain unconditionally stable schemes. It is found that the stability region is invariant for all LOD type of schemes and does not depend on the space dimensions. This type of observation has also been given by Fairweather *et al.* (1967) for a particular case. Now, the parameter θ can be used as a measure to compare the accuracy of various schemes in a particular family of scheme. For example, by taking the case of the two dimensional LOD schemes, we have the relation (cf. Fig. 2).

$$\theta = \tfrac{1}{2} < \tfrac{1}{2} + \frac{1}{12r} < \tfrac{1}{2} + \frac{1}{6r} < 1 + \frac{1}{6r}$$

for optimal LOD scheme, Mitchell and Fairweather scheme, Peaceman and Rachford scheme and Hubbard scheme, respectively, for a fixed value of the mesh ratio r. This shows that optimum LOD scheme is the best scheme in the family of the schemes with an order of accuracy $O(k + h^2)$, but for $\theta = \frac{1}{2} + 1/(12r)$, we get Mitchell and Fairweather scheme of high accuracy $O(k^2 + h^4)$. For the two dimensional SOD schemes, the corresponding relation is $\theta = \frac{1}{2} < \frac{1}{2} + 1/(12r) < 1 + 1/(6r)$ for 'optimum'

SOD scheme, Mitchell and Fairweather scheme and Samarskii and Andreev scheme, respectively, with the corresponding comments about their merits (cf. Fig. 1).

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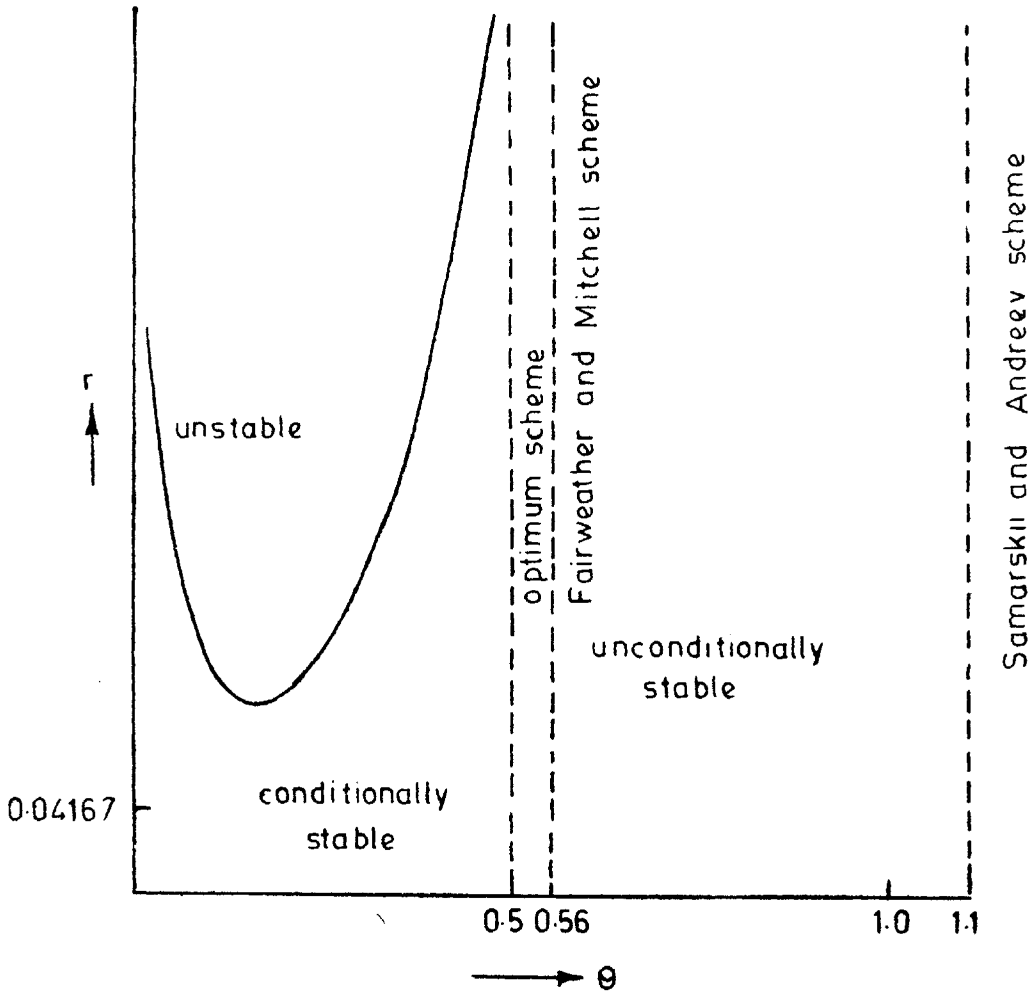

FIG. 1. Stability regions and the positions of various SOD schemes for $r = 1.5$.

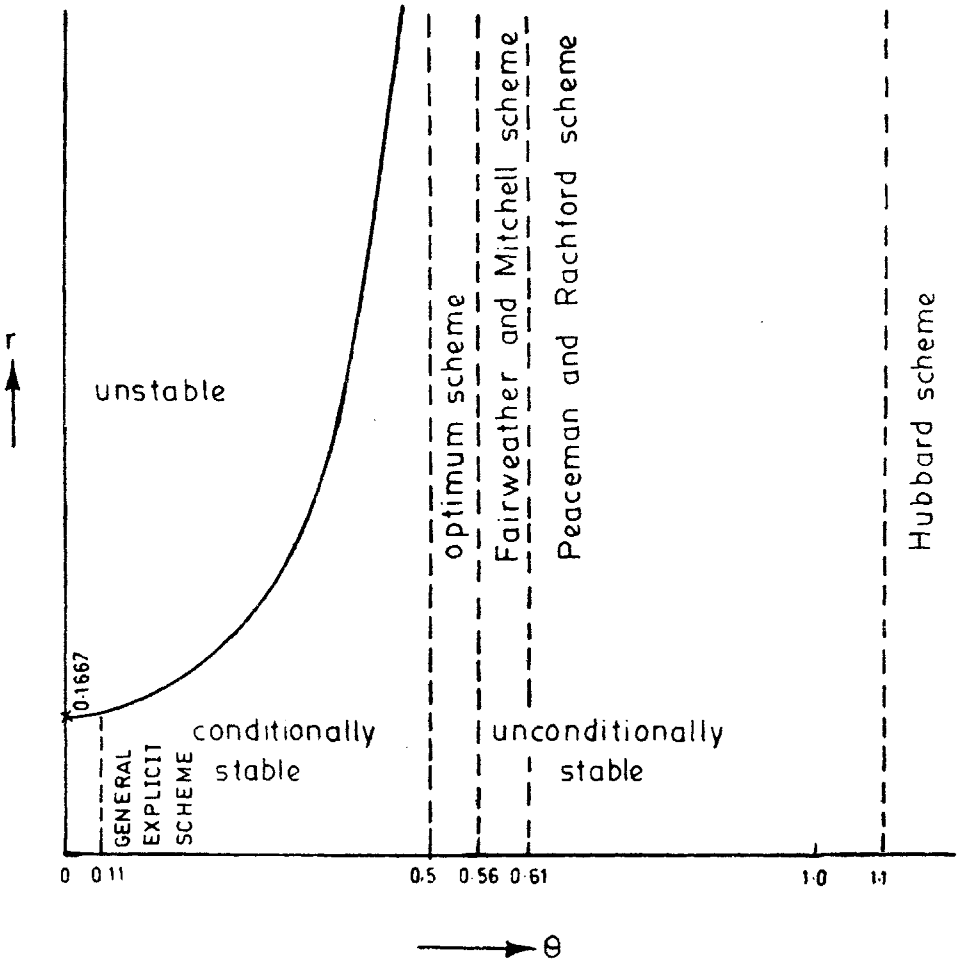

FIG. 2. Stability regions and the positions of various LOD schemes for $r = 1.5$.

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