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SYMMETRICAL VIBRATION OF A COMPOSITE CIRCULAR MEMBRANE FASTENED AT THE CIRCUMFERENCE

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In this note the problem of symmetrical vibration of a composite circular membrane has been studied. The membrane is formed by two concentric circles of which the inner circular region is composed of material of variable density σ_1 such that $\sigma_1 \propto r^m$ ($m = 0$ or any positive integer) whereas the outer annular region is composed of material of density σ_2 such that $\sigma_2 \propto (1/r)^n$ ($n = 0, 1, 2$ or any negative integer), where r is the radial distance from the common centre, distribution of matter being assumed continuous. The paper may be of practical importance in the designing of suitable vibrating membranes of a more complex structure for effective use in science, medicine and technology.

INTRODUCTION

The problem of vibration of circular membranes was first studied by Lord Rayleigh (1945) and subsequently by Sneddon (1946), Carslaw and Jaeger (1963) by use of double Fourier transform and Hankel transform techniques. The problem of transverse vibration of a piecewisely homogeneous composite circular membrane was studied by Vodička (1962). Subsequently Kundu (1967) studied the problem of a vibrating composite membrane consisting of two concentric circular regions, the inner region being composed of homogeneous material and the outer annular region having material of density varying inversely as the square of the distance from the centre. Recently Hutchinson and Niordson (1972) have in their paper made a very significant contribution on the designing of vibrating membranes of various shapes. In this note a new problem of symmetrical vibration of a composite circular membrane has been investigated, in which the inner concentric region and the outer annular region are assumed to be made up of materials of variable density σ_1 and σ_2 such that $\sigma = \sigma_1 = \sigma_0(r/a)^m$ for $0 \leqslant r \leqslant a$ ($m = 0$ or any positive integer) and $\sigma = \sigma_2 = \sigma_0(a/r)^n$ for $a \leqslant r \leqslant b$ ($n = 0, 1, 2$ or any negative integer), where a, b are the radii of the inner and outer circles and σ_0 is the density on the common boundary $r = a$. In the above assumption it is tacitly assumed that the membrane is sufficiently thin at the centre (origin) so that the density at $r = 0$ is practically negligible. The role of the membrane in practical use in different branches of science, technology and medicine is well known (Hunter 1962, Villars Fellx and Benedek 1974, Chapman 1970).

The artificial (industrial) membranes now being designed in the laboratories are far from being perfect and require for being a closer approximation to realities a much deeper knowledge of the dynamics of membranes of a more complex structure. It is believed that this paper which seeks to extend our knowledge of the theory of a vibrating composite membrane of a more complex structure may be of help in designing such membranes for a more effective use in science, medicine and technology.

FORMULATION

The equation of motion for symmetrical vibration of the entire membrane is

$$\frac{\partial^2 z}{\partial r^2} + \frac{1}{r}\frac{\partial z}{\partial r} = \frac{1}{c^2}\frac{\partial^2 z}{\partial t^2} \quad ...(1)$$

where $z = z(r, t)$ is the transverse displacement and

$$c^2 = \frac{T}{\sigma}. \quad ...(2)$$

For normal solution we assume

$$z = Z(r)\, e^{i\omega t}. \quad ...(3)$$

Substituting in (1) we obtain

$$\frac{d^2 Z}{dr^2} + \frac{1}{r}\frac{dZ}{dr} + \frac{\omega^2 \sigma}{T} Z = 0. \quad ...(4)$$

For the region $0 \leqslant r \leqslant a$, $Z = Z_1$ and $\sigma = \sigma_1 = \sigma_0 (r/a)^m$;

we have

$$\frac{d^2 Z_1}{dr^2} + \frac{1}{r}\frac{dZ_1}{dr} + s_1^2\, r^m Z_1 = 0 \quad ...(5)$$

where

$$s_1^2 = \frac{\sigma_0}{T}\,\omega^2 a^{-m}.$$

Substituting $\lambda = r^{(m+2)/2}$, eqn. (5) reduces to the form,

$$\frac{d^2 Z_1}{d\lambda^2} + \frac{1}{\lambda}\frac{dZ_1}{d\lambda} + s_2^2\, Z_1 = 0$$

yielding as its general solution

$$Z_1 = AJ_0(s_2 r^{(m+2)/2}) + BY_0(s_2 r^{(m+2)/2}) \quad ...(6)$$

where A and B are constants of integration and

$$s_2 = \frac{2s_1}{m+2}. \quad ...(7)$$

For the annular region $a \leqslant r \leqslant b$, $Z = Z_2$, $\sigma = \sigma_0(a/r)^n$

we shall similarly obtain from (1) as the complete solution

$$Z_2 = CJ_0(\beta_2 r^{(2-n)/2} + DY_0(\beta_2 r^{(2-n)/2} \quad \ldots(8)$$

where

$$\beta_2 = \frac{2\beta_1}{(2-n)}. \quad \ldots(9)$$

and

$$\beta_1^2 = \frac{\sigma_0 \omega^2 a^n}{T}. \quad \ldots(10)$$

Our problem now reduces to finding the values of the constants A, B, C, D in relations (6) and (8) for the two regions satisfying the following boundary/initial conditions:

(i) $Z_1(0)$ is finite, (ii) $Z_1(a) = Z_2(a)$

(iii) $Z_1'(a) = Z_2'(a)$, (iv) $Z_2(b) = 0$.

Consistent with condition (i) we have

$$B = 0 \text{ in } (6)$$

and consequently

$$Z_1 = AJ_0(s_2 r^{(m+2)/2}). \quad \ldots(11)$$

Using the boundary conditions (ii) and (iv) we obtain two relations involving A, C and D. Eliminating D from these two relations we obtain

$$AJ_0(s_2 a^{(m+2)/2})\, Y_0(\beta_2 b^{(2-n)/2}) = C\,[J_0(\beta_2 a^{(2-n)/2})\, Y_0(\beta_2 b^{(2-n)/2}) - J_0(\beta_2 b^{(2-n)/2})\, Y_0(\beta_2 a^{(2-n)/2})]. \quad \ldots(12)$$

Also applying boundary conditions (iii) and (iv) and then eliminating D from the relations so obtained we have

$$AJ_1(s_2 a^{(m+2)/2})\, Y_0(\beta_2 b^{(2-n)/2}) = C\,[J_1(\beta_2 a^{(2-n)/2})\, Y_0(\beta_2 b^{(2-n)/2}) - Y_1(\beta_2 a^{(2-n)/2})\, J_0(\beta_2 b^{(2-n)/2})]. \quad \ldots(13)$$

Dividing (12) by (13) we derive

$$J_0(s_2 a^{(m+2)/2})\,[J_1(\beta_2 a^{(2-n)/2})\, Y_0(\beta_2 b^{(2-n)/2}) - Y_1(\beta_2 a^{(2-n)/2})\, J_0(\beta_2 b^{(2-n)/2})]$$
$$= J_1(s_2 a^{(m+2)/2})\,[J_0(\beta_2 a^{(2-n)/2})\, Y_0(\beta_2 b^{(2-n)/2}) - J_0(\beta_2 b^{(2-n)/2})\, Y_0(\beta_2 a^{(2-n)/2})] \quad \ldots(14)$$

which is the frequency equation.

PARTICULAR CASES

Case 1 : $m = n = 0$, $a = b$ — In this case we have a complete circular membrane of uniform density σ_0 with fixed edges in which,

$$s_2 = s_1 = \frac{\omega}{c} = K \text{ (say) and } \beta_2 = \beta_1 = \frac{\omega}{c} = K.$$

Then the frequency equation (14) takes the form

$$J_0(Ka) = 0 \qquad \dots(15)$$

which is a well-known result (Ghosh 1975).

Case 2 : $m = n = 0$, $a \neq b$ — In this case we have a case of circular annulus defined by $a \leqslant r \leqslant b$ of continuous (uniform) density σ_0.

Assuming its edges $r = a$ and $r = b$ to be fixed, we obtain in this case

$$\beta_2 = \beta_1 = \frac{\omega}{c} = K \text{ (say)}$$

$$s_2 = s_1 = \frac{\omega}{c} = K$$

and $\qquad J_0(Ka) = 0$ [vide result (11) above].

We then derive from (14) the frequency equation in the form

$$J_0(Ka)\, Y_0(Kb) - J_0(Kb)\, Y_0(Ka) = 0 \qquad \dots(16)$$

which is a well-known result (Ghosh 1975).

Case 3 — A non-homogeneous complete membrane of density

$$\sigma = \sigma_0 \left(\frac{a}{r}\right)^n.$$

To derive this case, we assume $m = -n$ (n negative) and

$$a = b, \text{ then } s_2 = \frac{2s_1}{2-n},\ s_1 = \frac{\omega}{c}\, a^{n/2} = Ka^{n/2},$$

$$\beta_2 = \frac{2\beta_1}{2-n},\ \beta_1 = Ka^{n/2} = s_1.$$

$$\therefore \qquad \beta_2 = s_2 = \frac{2Ka^{n/2}}{(2-n)}.$$

Then the frequency equation (14) reduces to the form,

$$J_0\left(\frac{2Ka}{2-n}\right)\left[J_1\left(\frac{2Ka}{2-n}\right) Y_0\left(\frac{2Ka}{2-n}\right) - Y_1\left(\frac{2Ka}{2-n}\right) J_0\left(\frac{2Ka}{2-n}\right)\right] = 0.$$

or $$\frac{2-n}{\pi Ka} J_0\left(\frac{2Ka}{2-n}\right) = 0. \qquad \dots(17)$$

by virtue of the Wronskian relation

$$J_1(Z)\, Y_0(Z) - J_0(Z)\, Y_1(Z) = \frac{2}{\pi Z}.$$

Since $n \neq 2$ in our above analysis, we derive the frequency equation

$$J_0\left(\frac{2Ka}{2-n}\right) = 0. \qquad \ldots(18)$$

If x_m be a positive zero of $J_0(x)$ where $m = 1, 2, 3, \ldots$ give the ranking to the infinite number of such zeros, it follows from the above that

$$\omega_m = \frac{(2-n)}{2} \frac{x_m}{a} \sqrt{\frac{T}{\sigma_0}} \qquad \ldots(19)$$

where

$$K_m = \frac{\omega_m}{c}.$$

From (19) it is evident that no real value of the radial frequency would exist for any positive integral values of $n > 2$.

Numerical result—Take $a = 10$ cm. then the cyclic frequency $\omega_1/2\pi\,(=\alpha_n(T/\sigma_0)^{1/2})$ of the fundamental mode of vibration of a non-homogeneous complete circular membrane of radius 10 cm. and variable density $\sigma = \sigma_0(a/r)^n$ is shown in Table I for different admissible values of n.

TABLE I

Fundamental frequency: $\omega_1/2\pi = \alpha_n(T/\sigma_0)^{1/2}$

n	α_n
0	0.038
1	0.019
-2	0.077
-3	0.096
-4	0.115
-5	0.134

Case 4 : $m = 0$; $n = 1$ — In this case

$$s_2 = s_1 = \frac{\omega}{c} = K \text{ (say)};\ \beta_2 = 2\beta_1 = 2Ka^{1/2}.$$

Substituting in (14) we obtain the frequency equation in the form

$$J_0(Ka)\left[J_1(2Ka)\,Y_0\left(\sqrt{\frac{b}{a}}\,.\,2Ka\right) - Y_1(2Ka)\,J_0\left(\sqrt{\frac{b}{a}}\,.\,2Ka\right)\right]$$

$$= J_1(Ka)\left[J_0(2Ka)\,Y_0\left(\sqrt{\frac{b}{a}}\,.\,2Ka\right) - J_0\left(\sqrt{\frac{b}{a}}\,.\,2Ka\right)Y_0(2Ka)\right]. \qquad ...(20)$$

Case 5 — When $m = 0$, $n = 2$, $\sigma = \sigma_1 = \sigma_0$ (constant) and $\sigma = \sigma_2 = \sigma_0(a/r)^2$, the above method of solution fails in view of relation (9). But in this case eqn. (4) takes the form

$$r^2\,\frac{d^2Z_2}{dr^2} + r\,\frac{dZ_2}{dr} + a^2\alpha^2 Z_2 = 0$$

where $\alpha^2 = \frac{\omega^2\sigma_0}{T}$, which is homogeneous and solvable. Kundu (1967) has solved this case and has presented for the composite membrane the following frequency equation.

$$J_0(a\alpha)\cos\left(a\alpha\log\frac{a}{b}\right) + J_1(a\alpha)\sin\left(a\alpha\log\frac{a}{b}\right) = 0. \qquad ...(21)$$

Case 6 : $m \neq 0$, $n =$ any negative integer — When $m = 1$, $n = -4$, $s_2 = 2Ka^{-1/2}/3$, $\beta_2 = Ka^{-2}/3$ and the frequency equation (14) reduces to the form

$$J_0\left(\frac{2Ka}{3}\right)\left[J_1\left(\frac{Ka}{3}\right)Y_0\left(\tfrac{1}{3}\,Kb\,.\,\frac{b^2}{a^2}\right) - Y_1\left(\frac{Ka}{3}\right)J_0\left(\tfrac{1}{3}\,Kb\,.\,\frac{b^2}{a^2}\right)\right]$$

$$= J_1\left(\frac{2Ka}{3}\right)\left[J_0\left(\frac{Ka}{3}\right)Y_0\left(\tfrac{1}{3}\,Kb\,.\,\frac{b^2}{a^2}\right) - J_0\left(\tfrac{1}{3}\,Kb\,.\,\frac{b^2}{a^2}\right)Y_0\left(\frac{Ka}{3}\right)\right]. \qquad ...(22)$$

TABLE II

Cyclic frequency: $\omega/2\pi = \alpha_{m,n}(T/\sigma_0)^{1/2}$

m	n	$\alpha_{m,n}$
1	− 1	0.026
1	− 2	0.023
1	− 3	0.031
1	− 4	0.018
2	− 5	0.016

Numerical result — Take $a = 6$ cm., $b = 12$ cm. Then solving the frequency equation (22) we obtain $K = 0.115201$ (approx.), and, therefore, from relation (10) cyclic frequency $\omega/2\pi = 0.018\,(T/\sigma_0)^{1/2}$. Similarly for different values of m, n we obtain after solving the frequency equation (14) for $a = 6$ cm., $b = 12$ cm. different values of K and hence different cyclic frequencies $\omega/2\pi$ of vibration. These are tabulated in Table II.

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