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ABSOLUTE RIESZ SUMMABILITY SEQUENCES FOR FOURIER SERIES*

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In this paper some absolute Riesz summability sequences connected with Fourier series and its conjugate series of a Lebesgue integrable function are obtained. The main result is as follows:

$$\Sigma\{P_n\, l_n\}/L_n < \infty \Rightarrow \{P_n\, A_n(x)\} \in |\, R, L_{n-1}, 1\,|,$$

where $\Sigma A_n(x)$ is Fourier series, $L_n = l_0 + l_1 + \ldots + l_n$ and $\{L_n\}$ is strictly increasing sequence of non-negative terms.

1. DEFINITIONS AND NOTATIONS

Let $\{S_n\}$ be a given infinite sequence and $\{L_n\}$ be a strictly increasing sequence of non-negative terms tending to infinity as $n \to \infty$. We write

$$t_n = L_n^{-1} \sum_{m=0}^{n} l_m S_m \; : \; (L_n = l_0 + l_1 + \ldots + l_n).$$

t_n is said to be Riesz mean of the sequence $\{S_n\}$ of type L_n and order unity. If $t_n \to s$ as $n \to \infty$, then we say that the sequence $\{S_n\}$ is summable $(R, L_{n-1}, 1)$ to the sum s, and also if $\{t_n\} \in BV$, then the sequence $\{S_n\}$ is said to be absolutely summable $(R, L_{n-1}, 1)$ or symbolically $\{S_n\} \in |\, R, L_{n-1}, 1\,|$.

Now

$$Dt_n = t_n - t_{n+1}, \text{ hence the sequence } \{S_n\} \in |\, R, L_{n-1}, 1\,|$$

if

$$\sum \equiv \sum_{n=1}^{\infty} \Big|\; D\,(L_n^{-1}) \sum_{m=0}^{n} l_m S_m - (S_{n+1} l_{n+1}\, L_{n+1}^{1} \;\Big| < \infty.$$

Let $f(x)$ be a 2π-periodic function and L-integrable in $(-\pi, \pi)$. Without loss of generality we assume that constant term of its Fourier series is zero so that

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$$f(t) \sim \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) \equiv \sum_{n=1}^{\infty} A_n(t)$$

and conjugate to this series is

$$\sum_{n=1}^{\infty} (b_n \cos nt - a_n \sin nt) \equiv \sum_{n=1}^{\infty} B_n(t).$$

Throughout this paper, we use the following notations:

$$\varphi(t) = \tfrac{1}{2}\{f(x + t) + f(x - t)\} \qquad \ldots(1)$$

$$\psi(t) = \tfrac{1}{2}\{f(x + t) - f(x - t)\} \qquad \ldots(2)$$

$$F(t) = \int_0^t \varphi(u)\, du \qquad \ldots(3)$$

$$\varphi_1(t) = \frac{2}{\left(\log \frac{k}{t}\right)} \int_0^t \varphi(u) \cot \frac{u}{2}\, du. \qquad \ldots(4)$$

2. Introduction

Concerning ordinary Riesz summability of a sequence connected with Fourier series, following theorems are known.

Theorem A (Mohanty and Rao 1968) — If $\varphi_1(t) \in BV(0, \pi)$ then the sequence $\{nA_n(x)\} \in (R, \log n, 1)$ to the value zero.

Theorem B (Chandra 1971) — Let the sequence $\{P_n\}$ and $\{L_n\}$ satisfy the condition

$$\sum_{n=1}^{\infty} \frac{\{P_n\, l_n\}}{\{n\, L_n\}} < \infty \qquad \ldots(5)$$

then

$$\varphi(t) \in BV(0, \pi) \Rightarrow \{P_n\, A_n(x)\} \in \mid R, L_{n-1}, 1 \mid . \qquad \ldots(6)$$

The purpose of this paper is to obtain the absolute Riesz summability of sequences connected with Fourier series and its conjugate series without imposing any restrictions on the generating function. The main results of this paper are Theorems 1 and 2, which are of very general character and lead to many interesting results concerning the absolute Riesz summability, absolute Cesàro and absolute harmonic summability of sequences connected with Fourier series of Lebesgue integrable functions. We first prove the following :

Theorem 1 — Let the sequences $\{P_n\}$ and $\{L_n\}$ satisfy the condition

$$\sum_{n=1}^{\infty} \frac{\{P_n l_n\}}{L_n} < \infty; \; P_n \geqslant 0, \, n = 1, 2, \ldots \qquad \ldots(7)$$

then

$$\{P_n A_n(x)\} \in | \, R, L_{n-1}, 1 \, | \, .$$

Theorem 2 — If the sequences $\{P_n\}$ and $\{L_n\}$ satisfy (7), then

$$\{P_n B_n(x)\} \in | \, R, L_{n-1}, 1 \, | \, .$$

3. Lemmas

We shall use the following Lemmas for the proof of the theorems.

Lemma 1 (Dikshit 1965) — The sequence $\{S_n\} \in | \, R, \exp \left\{\frac{n}{\log n}\right\}, 1 \, |$, implies $\{S_n\} \in | \, N, (n+1)^{-1} \, |$.

Lemma 2 (Dikshit 1965) — The sequence

$$\{S_n\} \in | \, R, \exp (n^{\alpha}), 1 \, | , (0 < \alpha \leqslant 1) \text{ implies } \{S_n\} \in | \, C, \delta \, | ,$$

for $\delta \geqslant 1 - \alpha$.

4. Proof of the Theorems

Proof of Theorem 1 — We have

$$A_n(x) = \frac{2}{\pi} \int_0^{\pi} \varphi(t) \cos nt \, dt$$

$$= \frac{2}{\pi} \Big[\cos nt \, F(t) \Big]_0^{\pi} - \frac{2}{\pi} \int_0^{\pi} F(t) \, n \sin nt \, dt$$

$$= -\frac{2}{\pi} \int_0^{\pi} F(t) \, . \, n \sin nt \, dt, \text{ (since } F(\pi) = 0).$$

The sequence $\{S_n\} \in |R, L_{n-1}, 1|$, if

$$\sum_{n=1}^{\infty} \left| D(L_n^{-1}) \sum_{m=1}^{n} l_m S_m \right| < \infty \qquad \ldots(8)$$

and

$$\sum_{n=1}^{\infty} \left| S_n l_n L_n^{-1} \right| < \infty. \qquad \ldots(9)$$

Now $F(t)$ being an integral, is a function of bounded variation in $(0, \pi)$, so that $F(t)$ may be written as difference of two positive, monotonic increasing functions in $(0, \pi)$. Let $F(t) = \varphi_2(t) - \varphi_3(t)$ such that $\varphi_2(t)$ and $\varphi_3(t)$ are positive, bounded and non-decreasing in $(0, \pi)$. Hence

$$\int_0^{\pi} F(u) \sin nu\, du = \int_0^{\pi} \varphi_2(u) \sin nu\, du - \int_0^{\pi} \varphi_3(u) \sin nu\, du$$

$$= \varphi_2(\pi - 0) \int_t^{\pi} \sin nu\, du + \varphi_2(+0) \int_0^{t} \sin nu\, du$$

$$- \varphi_3(\pi - 0) \int_{t_1}^{\pi} \sin nu\, du + \varphi_3(+0) \int_0^{t_1} \sin nu\, du$$

$$(0 \leqslant t \leqslant \pi,\ 0 \leqslant t_1 \leqslant \pi).$$

Now by (8) and (9) to prove the theorem, it is sufficient to show that

$$\sum\nolimits_1 \equiv \sum_{n=1}^{\infty} \left| D(L_n^{-1}) \sum_{m=1}^{n} P_m l_m \int_t^{\pi} n \sin nu\, du \right| = O(1) \qquad \ldots(10)$$

$$\sum\nolimits_2 \equiv \sum_{n=1}^{\infty} \left| D(L_n^{-1}) \sum_{m=1}^{n} P_m l_m \int_0^{t} n \sin nu\, du \right| = O(1) \qquad \ldots(11)$$

and

$$\sum\nolimits_3 \equiv \sum_{n=1}^{\infty} \left| P_n l_n L_n^{-1} \int_t^{\pi} n \sin nu\, du \right| = O(1) \qquad \ldots(12)$$

$$\sum\nolimits_4 \equiv \sum_{n=1}^{\infty} \left| P_n l_n L_n^{-1} \int_0^{t} n \sin nu\, du \right| = O(1) \qquad \ldots(13)$$

uniformly in $0 < t \leqslant \pi$.

We have

$$\sum_2 = \sum_{n=1}^{\infty} \left| D(L_n^{-1}) \sum_{m=1}^{n} P_m l_m(\cos nt - 1) \right|$$

$$= O\left\{ \sum_{m=1}^{\infty} P_m l_m \sum_{n=m}^{\infty} (L_n^{-1} - L_{n+1}^{-1}) \right\}$$

$$= O\left\{ \sum_{m=1}^{\infty} \frac{P_m l_m}{L_m} \right\} = O(1).$$

Also

$$\sum_4 = O\left\{ \sum_{n=1}^{\infty} \frac{P_n l_n}{L_n} \right\} = O(1).$$

Both bounds being uniform in $0 < t \leqslant \pi$, Σ_1 and Σ_3 are disposed of in similar ways. This completes the proof of Theorem 1.

Proof of Theorem 2 — We have

$$B_n(x) = \frac{2}{\pi} \int_0^{\pi} \psi(t) \sin nt \, dt$$

$$= -\frac{2}{\pi} \int_0^{\pi} \psi_1(t)\, n \cos nt \, dt \; : \; \left\{ \psi_1(t) \equiv \int_0^{t} \psi(u)\, du \right\}.$$

Hence the proof runs on the lines of the proof of Theorem 1.

5. Some Results

In this section we give those results which are suggested by Theorems 1 and 2. Following results are stated for Fourier series but they also hold for its conjugate series as suggested by Theorem 2.

Corollary 1 — The sequence $\{q_n A_n(x)\} \in |R, (\exp(n^\alpha)), 1|$ provided that

$$\sum_{n=1}^{\infty} \frac{q_n}{n^{1-\alpha}} < \infty \; (0 < \alpha \leqslant 1). \qquad \ldots(12)$$

Here we note that $\dfrac{l_n}{L_n} = \dfrac{\exp(n^\alpha) - \exp(n-1)^\alpha}{\exp(n^\alpha)} = O(n^{\alpha-1})$.

Corollary 2 — The sequence $\{q_n A_n(x)\} \in |R, \exp(\log n)^c, 1|$ provided that

$$\sum_{n=1}^{\infty} \frac{q_n}{n(\log n)^{1-c}} < \infty. \qquad \ldots(13)$$

In this case we have $\dfrac{l_n}{L_n} = \dfrac{\exp\{(\log n)^c\} - \exp\{(\log \overline{n-1})^c\}}{\exp\{(\log n)^c\}}$

$$= O\left\{\frac{1}{n(\log n)^{1-c}}\right\}.$$

Corollary 3 — For $b > 0$ the sequence $\{P_n A_n(x)\} \in |R, \exp\{n(\log n)^{-b}\}, 1|$, provided that

$$\sum_{n=2}^{\infty} \frac{P_n}{\{(\log n)^b\}} < \infty. \qquad \ldots(14)$$

PROOF : Since in this case $l_n/L_n = O\{(\log n)^{-b}\}$, hence the result follows by Theorem 1.

Theorem 3 — The sequence $\{P_n A_n(x)\} \in \left|N, \frac{1}{n+1}\right|$, provided that

$$\sum_{n=2}^{\infty} \frac{P_n}{(\log n)} < \infty. \qquad \ldots(15)$$

Theorem 4 — The sequence $\{q_n A_n(x)\} \in |c, \alpha|$, provided that

$$\sum_{n=1}^{\infty} \frac{q_n}{n^{1-\alpha}} < \infty, \; 0 < \alpha \leqslant 1. \qquad \ldots(16)$$

PROOF OF THEOREMS 3 and 4 : Since $\{P_n A_n(x)\} \in \left| R, \exp\left(\frac{n}{\log n}\right), 1 \right|$, hence Theorem 3 follows in view of Lemma 1. Also $\{q_n A_n(x)\} \in |R, \exp(n^{\alpha}), 1|$, hence the proof of Theorem 4 follows by Lemma 2.

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