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TWO ASYMMETRICALLY PLACED GRIFFITH CRACKS IN AN INFINITE ELASTIC STRIP

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The distribution of stress in the neighbourhood of two Griffith cracks located asymmertically in an infinitely long elastic strip is considered. It is assumed that the edges of the strip are stress free, and the cracks are opened by an internal pressure varying along its length. The expressions for the quantities of physical interest like stress intensity factors, shape of deformed cracks and the crack energy are derived up to the order of δ^{-6}, where 2δ denotes the thickness of the strip.

1. INTRODUCTION

Recently Calhoun and Lowengrub (1974) have studied the problem of determining stress distribution in the neighbourhood of a Griffith crack placed asymmetrically in an infinitely long elastic strip. Use of Fourier transforms reduces the problem to solving simultaneous set of dual integral equations containing trigonometric kernel and certain weight functions. These equations are then reduced to a pair of Fredholm integral equations which are solved by iterative procedure, valid for the ratio of the thickness of the strip to radius of the crack which is greater than unity.

In this paper, we have determined the stress distribution in the neighbourhood of two Griffith cracks placed asymmetrically in an infinitely long elastic strip. Use of Fourier transforms reduces this problem to solving simultaneous set of triple integral equations containing trigonometric kernels and certain weight functions. The technique of finite Hilbert transforms used by Srivastava and Lowengrub (1968) to solve a set of triple integral equations with trigonometric kernels and certain weight functions, is extended to solve the above simultaneous set of triple integral equations. The finite Hilbert transform reduces this set to a set of simultaneous Fredholm integral equations of the second kind. This set is now solved by iterative procedure up to the order of δ^{-6}, where 2δ denotes thickness of the strip. The quantities of physical interest are expressed in closed forms in terms of elliptic functions of the first and second kinds.

2. THE BASIC EQUATIONS

The material of the strip is supposed to occupy the region defined by $-\infty < x < \infty$, $-\delta \leqslant y \leqslant \delta$ with the cracks in the interior of the material on the

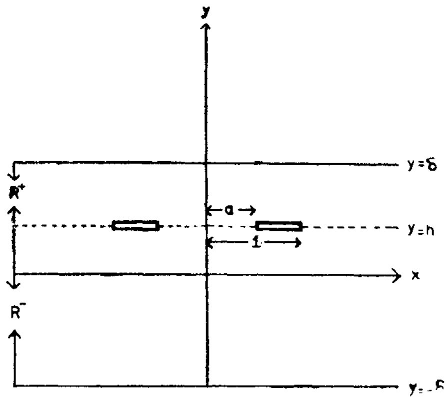

FIG. 1. Two Griffith cracks parallel to the boundary.

line $y = h$, $a < |x| < 1$ (cf. Fig. 1). In rectangular coordinates (x, y), the displacement and stress components are respectively u_x, u_y and σ_{xx}, σ_{yy}, σ_{xy}. The edges of the strip are assumed to be free from stress.

We denote by R^+ and R^- the regions defined by $h \leqslant y \leqslant \delta$ and $-\delta \leqslant y \leqslant h$ respectively. If the cracks are subjected to an internal pressure $p(x)$ varying along the length of cracks, the boundary conditions take the form:
On $|y| = \delta$

$$\sigma_{yy} = \sigma_{xy} = 0, \; -\infty < x < \infty \qquad \text{...(2.1)}$$

On $y = h$

$$\sigma^+_{yy}(x, h+) = \sigma^-_{yy}(x, h-) = \sqrt{\frac{2}{\pi}}\, p(x), \; a < |x| < 1 \qquad \text{...(2.2)}$$

$$\sigma^+_{xy}(x, h+) = \sigma^-_{xy}(x, h-) = 0, \; a < |x| < 1. \qquad \text{...(2.3)}$$

For the region unoccupied by the cracks we have following continuity conditions :

$$\left.\begin{aligned} \sigma^+_{yy}(x, h+) &= \sigma^-_{yy}(x, h-) \\ \sigma^+_{xy}(x, h+) &= \sigma^-_{xy}(x, h-) \\ u^+_x(x, h+) &= u^-_x(x, h-) \\ u^+_y(x, h+) &= u^-_y(x, h-) \end{aligned}\right\}, \; 0 < |x| < a, \; |x| > 1 \qquad \text{...(2.4)}$$

where σ^+, u^+, and σ^-, u^- are stress and displacement components in R^+ and R^- respectively.

Calhoun and Lowengrub (1974) have shown that the components of displacement and stress can be expressed in terms of a stress function $G(\xi, y)$ as follows

$$Eu_x^{\pm}(x, y) = i(1+\eta)\,\mathcal{F}^*\left[\xi^{-1}\left\{(1-\eta)\frac{\partial^2}{\partial y^2}G^{\pm} + \eta\xi^2 G^{\pm}\right\}; \xi \to x\right] \quad \ldots(2.5)$$

$$Eu_y^{\pm}(x, y) = (1+\eta)\,\mathcal{F}^*\left[\xi^{-2}\left\{(1-\eta)\frac{\partial^3}{\partial y^3}G^{\pm} + (\eta-2)\frac{\partial}{\partial y}G^{\pm}\right\}; \xi \to x\right] \quad \ldots(2.6)$$

$$\sigma_{xx}^{\pm}(x, y) = \mathcal{F}^*\left[\frac{\partial^2}{\partial y^2}G^{\pm}; \xi \to x\right] \quad \ldots(2.7)$$

$$\sigma_{yy}^{\pm}(x, y) = -\,\mathcal{F}^*\left[\xi^2 G^{\pm}; \xi \to x\right] \quad \ldots(2.8)$$

$$\sigma_{xy}^{\pm}(x, y) = i\,\mathcal{F}^*\left[\xi\frac{\partial}{\partial y}G^{\pm}; \xi \to x\right] \quad \ldots(2.9)$$

where $G^{\pm}(\xi, y)$ satisfies the equation

$$\left(\frac{\partial^2}{\partial y^2} - \xi^2\right)^2 G^{\pm} = 0 \quad \ldots(2.10)$$

and $\mathcal{F}^*$ denotes the Fourier inverse transform.

The appropriate solution of (2.10) satisfying the conditions (2.1) is given by

$$G^{\pm}(\xi, y) = A^{\pm}(\xi)\left[\sinh \xi(y \mp \delta) - \xi(y \mp \delta)\cosh \xi(y \mp \delta)\right] + B^{\pm}(\xi)(y \mp \delta)\sinh \xi(y \mp \delta). \quad \ldots(2.11)$$

We observe from (2.2), (2.3) and first two equations of (2.4) that $\sigma_{yy}^{+} = \sigma_{yy}^{-}$ and $\sigma_{xy}^{+} = \sigma_{xy}^{-}$ for all x. Thus, we can determine two constants of (2.11) from these two conditions. From the remaining boundary conditions, it follows that we must determine remaining two constants of (2.11) which satisfy the following system of triple integral equations

$$\mathcal{F}_s\left[\xi\{C_{11}(\xi)A^-(\xi) + C_{12}(\xi)B^-(\xi)\}; x\right] = 0, \ a \leqslant x \leqslant 1 \quad \ldots(2.12)$$

$$\mathcal{F}_c\left[\xi^2\{C_{21}(\xi)A^-(\xi) + C_{22}(\xi)B^-(\xi)\}; x\right] = \sqrt{\frac{2}{\pi}}\,p(x), \ a \leqslant x \leqslant 1 \quad \ldots(2.13)$$

$$\mathcal{F}_s\left[\frac{C_{31}(\xi)A^-(\xi) + C_{32}(\xi)B^-(\xi)}{\sinh^2 \xi(h-\delta) - \xi^2(h-\delta)^2}; x\right] = 0, \ 0 \leqslant x \leqslant a, \ 1 < x < \infty \quad \ldots(2.14)$$

$$\mathcal{F}_c\left[\xi\left\{\frac{C_{41}(\xi)A^-(\xi) + C_{42}(\xi)B^-(\xi)}{\sinh^2 \xi(h-\delta) - \xi^2(h-\delta)^2}\right\}; x\right] = 0, \ 0 \leqslant x \leqslant a, \ 1 < x < \infty \quad \ldots(2.15)$$

where C_{ij} are given by

$$C_{11}(\xi) = -\xi\theta \sinh\theta,\ C_{12}(\xi) = \sinh\theta + \theta\cosh\theta$$

$$C_{21}(\xi) = \sinh\theta - \theta\cosh\theta,\ C_{22}(\xi) = \xi^{-1}\theta\sinh\theta$$

$$C_{31}(\xi) = 4\xi^3\delta(h-\delta)\sinh\theta + \xi\theta\,[\cosh(\alpha - 2\xi\delta) - \cosh\theta]$$

$$C_{32}(\xi) = (1 - 4\alpha\xi\delta)\cosh\theta - \cosh(\alpha - 2\xi\delta) + \theta\sinh(\alpha - 2\xi\delta) - (\alpha - 2\xi\delta)\sinh\theta$$

$$C_{41}(\xi) = (4\alpha\xi\delta - 1)\cosh\theta + \cosh(\alpha - 2\xi\delta) + \theta\sinh(\alpha - 2\xi\delta) - (\alpha - 2\xi\delta)\sinh\theta$$

$$C_{42}(\xi) = -4\alpha\delta\sinh\theta + \xi^{-1}\theta\,[\cosh(\alpha - 2\xi\delta) - \cosh\theta]$$

with $\theta = \xi(h+\delta)$, $\alpha = \xi(h-\delta)$. In the above derivation, we have assumed $p(x)$ to be an even function of x on the interval $a \leqslant x \leqslant 1$ and the definitions

$$\mathcal{F}_c\,[f(\xi);\, x] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(\xi)\cos\xi x\, d\xi$$

$$\mathcal{F}_s\,[f(\xi);\, x] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(\xi)\sin\xi x\, d\xi.$$

3. Solution of the Simultaneous Triple Integral Equations

In order to simplify the analysis it is desirable at this stage to define two new functions ψ_1 and ψ_2 as follows

$$\psi_1(\xi) = \frac{C_{31}(\xi)\,A^-(\xi) + C_{32}(\xi)\,B^-(\xi)}{\sinh^2\xi(h-\delta) - \xi^2(h-\delta)^2} \qquad \ldots(3.1)$$

$$\psi_2(\xi) = \xi\,\frac{C_{41}(\xi)\,A^-(\xi) + C_{42}(\xi)\,B^-(\xi)}{\sinh^2\xi(h-\delta) - \xi^2(h-\delta)^2} \qquad \ldots(3.2)$$

so that the set of simultaneous triple integral eqns. (2.12) to (2.15) takes the form

$$\left.\begin{aligned}\mathcal{F}_s\,[\psi_1(\xi);\, x] &= 0\\ \mathcal{F}_c\,[\psi_2(\xi);\, x] &= 0\end{aligned}\right\}\ 0 \leqslant x \leqslant a,\ 1 \leqslant x < \infty \qquad \ldots(3.3)$$

$$\left.\begin{aligned}&\mathcal{F}_s\,[\xi\,\{(N_{11}(\xi) - 1)\,\psi_1(\xi) + N_{12}\psi_2(\xi)\};\, x] = 0\\ &\mathcal{F}_c\,[\xi\,\{N_{21}(\xi)\,\psi_1(\xi) + (N_{22}(\xi) + 1)\,\psi_2(\xi)\};\, x] = \sqrt{\frac{2}{\pi}}\,p(x)\end{aligned}\right\}\ (a \leqslant x \leqslant 1) \qquad \ldots(3.4)$$

where

$$N_{11}(\xi) = 1 + \frac{M_{11}(\xi)}{\sinh^2 2\xi\delta - (2\xi\delta)^2}, \quad N_{22}(\xi) = \frac{M_{22}(\xi)}{\sinh^2 2\xi\delta - (2\xi\delta)^2} - 1$$

$$\left. N_{ij} = \frac{M_{ij}(\xi)}{\sinh^2 2\xi\delta - (2\xi\delta)^2}, \text{ for } i \neq j, \; i = 1, 2; \; j = 1, 2 \right\} \qquad \ldots(3.5)$$

and

$$M_{11}(\xi) = 2\xi\delta(1 - 2\alpha\theta) - (\tfrac{1}{2} + \theta^2) \sinh 2\alpha + (\tfrac{1}{2} + \alpha^2) \sinh 2\theta$$
$$- \theta \cosh 2\alpha + \alpha \cosh 2\theta - \tfrac{1}{2} \sinh (4\xi\delta)$$
$$M_{22}(\xi) = 2\xi\delta(1 - 2\alpha\theta) + (\tfrac{1}{2} + \theta^2) \sinh 2\alpha - (\tfrac{1}{2} + \alpha^2) \sinh 2\theta$$
$$- \theta \cosh 2\alpha + \alpha \cosh 2\theta + \tfrac{1}{2} \sinh (4\xi\delta)$$
$$M_{12}(\xi) = - \xi \, [\alpha^2 \cosh 2\theta - \theta^2 \cosh 2\alpha + 4h\delta\xi^2]$$
$$M_{21}(\xi) = - \xi^{-2} M_{12}(\xi). \qquad \ldots(3.6)$$

We shall use the finite Hilbert transform technique of Srivastava and Lowengrub (1968) to reduce the above set of simultaneous triple integral equations to a pair of simultaneous Fredholm integral equations of the second kind. Let the trial solution be

$$\psi_1(\xi) = \xi^{-1} \int_a^1 t^{-1} g(t^2) (1 - \cos \xi t) \, dt \qquad \ldots(3.7)$$

$$\psi_2(\xi) = \xi^{-1} \int_a^1 h(t^2) \sin \xi t \, dt. \qquad \ldots(3.8)$$

Equations (3.3) are satisfied provided

$$\int_a^1 t^{-1} g(t^2) \, dt = 0, \qquad \int_a^1 h(t^2) \, dt = 0. \qquad \ldots(3.9)$$

The eqns. (3.4) reduce to following Abel type integral equations

$$\frac{2}{\pi} \int_a^1 \frac{th(t^2)}{t^2 - x^2} dt = \frac{2}{\pi} p(x) - \frac{2}{\pi} \left[\int_a^1 t^{-1} g(t^2) \int_0^\infty N_{21}(\xi) \cos \xi x (1 - \cos \xi t) \right.$$
$$\left. \times \, d\xi \, dt + \int_a^1 h(t^2) \int_0^\infty N_{22}(\xi) \sin \xi t \cos \xi x \, d\xi \, dt \right]$$

$$\frac{2}{\pi} \int_a^1 \frac{tg(t^2) \, dt}{t^2 - x^2} = \frac{2}{\pi} \left[\int_a^1 t^{-1} g(t^2) \int_0^\infty N_{11}(\xi) \sin \xi x (1 - \cos \xi t) \, d\xi \, dt \right.$$
$$\left. + \int_a^1 h(t^2) \int_0^\infty N_{12}(\xi) \sin \xi t \sin \xi x \, d\xi \, dt \right].$$

Inverting these Abel equations we get

$$h(x^2) = \phi_1(x) + \int_a^1 h(t^2)\, R_1(t, x)\, dt + \int_a^1 t^{-1} g(t^2)\, R_2(t, x)\, dt + l/X \quad \ldots(3.10)$$

$$g(x^2) = \int_a^1 t^{-1} g(t^2)\, R_3(t, x)\, dt + \int_a^1 h(t^2)\, R_4(t, x)\, dt + l_1/X \quad \ldots(3.11)$$

where l and l_1 are arbitrary constants to be determined from conditions (3.9), and

$$R_1(t, x) = \frac{2}{\pi}\, \mathcal{H}\left[\int_0^\infty N_{22}(\xi\delta) \sin \xi t \cos \xi x \, d\xi\right]$$

$$R_2(t, x) = \frac{2}{\pi}\, \mathcal{H}\left[\int_0^\infty N_{21}(\xi\delta) \cos \xi x (1 - \cos \xi t)\, d\xi\right]$$

$$R_3(t, x) = \frac{2}{\pi}\, \mathcal{H}\left[\int_0^\infty N_{11}(\xi\delta) \sin \xi x (1 - \cos \xi t)\, d\xi\right]$$

$$R_4(t, x) = \frac{2}{\pi}\, \mathcal{H}\left[\int_0^\infty N_{12}(\xi\delta) \sin \xi x \sin \xi t \, d\xi\right]$$

$$\phi_1(x) = -\frac{2}{\pi}\, \mathcal{H}[p(x)], \quad X(x) = \{(x^2 - a^2)(1 - x^2)\}^{1/2}.$$

Here the operator $\mathcal{H}$ is defined by

$$\mathcal{H}[f(y)] = \frac{2}{\pi} \int_a^1 \left\{\frac{(x^2 - a^2)(1 - y^2)}{(1 - x^2)(y^2 - a^2)}\right\}^{1/2} \cdot \frac{y f(y)\, dy}{y^2 - x^2}.$$

The elimination of arbitrary constants l and l_1 gives us following simultaneous Fredholm integral equations

$$h(x^2) = \phi(x) + \int_a^1 h(t^2)\, \Gamma_1(t, x)\, dt + \int_a^1 t^{-1} g(t^2)\, \Gamma_2(t, x)\, dt \quad \ldots(3.12)$$

$$g(x^2) = \int_a^1 t^{-1} g(t^2)\, \Gamma_3(t, x)\, dt + \int_a^1 h(t^2)\, \Gamma_4(t, x)\, dt \quad \ldots(3.13)$$

where

$$\phi(x) = \phi_1(x) - \frac{1}{FX} \int_a^1 \phi_1(y)\, dy \quad \ldots(3.14)$$

$$\Gamma_i(t, x) = R_i(t, x) - \frac{1}{FX}\int_a^1 R_i(t, y)\, dy, \ i = 1, 2 \qquad ...(3.15)$$

$$\Gamma_i(t, x) = R_i(t, x) - \frac{2a}{FX}\int_a^1 y^{-1}R_i(t, x)\, dy, \ i = 3, 4. \qquad ...(3.16)$$

Following Calhoun and Lowengrub (1974), we write $N_{ij}(s)$ in ascending powers of δ^{-1} as

$$N_{ij}(s) = q_{ij}^{(2)}(s) + q_{ij}^{(4)}(s)\,\delta^{-2} + q_{ij}^{(6)}(s)\,\delta^{-4} + O(\delta^{-6});\ s = \xi\delta$$

where different q_{ij} are given by eqns. (3.22) to (3.33) of Calhoun and Lowengrub (1974). The eqns. (3.15) and (3.16) will take following form after substituting N_{ij} and expanding sine and cosine functions in ascending powers of δ^{-1}. The calculations are further simplified by choosing h as our unit of length. This is equivalent to setting $h = 1$. The calculations for higher order of δ^{-1} can be carried similarly.

$$\Gamma_1(t, x) = \frac{2t}{\pi X}\left[\{I_0\delta^{-2} + I_1\delta^{-4} + I_2\,[t^2 + \tfrac{3}{2}(1 - a^2)]\,\delta^{-4}\}\,(E/F - x^2) + I_2\,\{3x^2(a^2 - x^2) - E(1 - 2a^2)/F - a^2\}\,\delta^{-4} + O(\delta^{-6})\right]$$

$$\Gamma_2(t, x) = \frac{2t^2}{\pi X}\left[\{I_3\delta^{-3} + I_4\delta^{-5} + I_5\,[t^2 + \tfrac{3}{2}(1 - a^2)]\,\delta^{-5}\}\,(E/F - x^2) + I_5\,\{6x^2(a^2 - x^2) - E(1 - 2a^2)/F - a^2\}\,\delta^{-5} + O(\delta^{-7})\right]$$

$$\Gamma_3(t, x) = \frac{2t^2}{\pi X}\left[\left\{I_6x^2\left\{\frac{1 + a^2}{2} - x^2\right\}\delta^{-4} + O(\delta^{-6})\right]$$

$$\Gamma_4(t, x) = \frac{2t}{\pi X}\left[x^2(I_7\delta^{-3} + I_8\delta^{-5})\left\{\frac{1 + a^2}{2} - x^2\right\} + I_9\left\{(x^2 + t^2) \times \left(\frac{1 + a^2}{2} - x^2\right) + .5\left(\frac{1 - a^2}{2}\right)^2\right\}\delta^{-5} + O(\delta^{-7})\right]$$

where $F(\pi/2, k) = F$ and $E(\pi/2, k) = E$; $k = \sqrt{(1 - a^2)}$ are complete elliptic integrals of the first and second kind, and

$$I_0 = \int_0^\infty s q_{22}^{(2)}(s)\, ds, \qquad I_1 = \int_0^\infty s q_{22}^{(4)}(s)\, ds$$

$$I_2 = -\frac{1}{3!}\int_0^\infty s^3 q_{22}^{(2)}(s)\, ds, \quad I_3 = \frac{1}{2!}\int_0^\infty s^2 q_{21}^{(2)}(s)\, ds$$

$$I_4 = \frac{1}{2!}\int_0^\infty s^2 q_{21}^{(4)}(s)\,ds, \qquad I_5 = -\frac{1}{4!}\int_0^\infty s^4 q_{21}^{(2)}(s)\,ds$$

$$I_6 = \frac{1}{2!}\int_0^\infty s^3 q_{11}^{(2)}(s)\,ds, \qquad I_7 = \int_0^\infty s^2 q_{12}^{(2)}(s)\,ds$$

$$I_8 = \int_0^\infty s^2 q_{12}^{(4)}(s)\,ds, \qquad I_9 = -\frac{1}{3!}\int_0^\infty s^4 q_{12}^{(2)}(s)\,ds.$$

The numerical values of above constants are obtained using tables given by Ling (1958) along with identities (3.35) to (3.37) of Calhoun and Lowengrub (1974). These are

$$I_0 = 2.285, \quad I_1 = -13.986, \quad I_2 = -8.549, \quad I_3 = 1.713,$$

$$I_4 = 14.862, \quad I_5 = -0.634, \quad I_6 = 11.60, \quad I_7 = 0.0,$$

$$I_8 = -20.277, \quad I_9 = 0.0.$$

The solutions of (3.12) and (3.13) are taken in the following form:

$$h(x^2) = h_0(x) + h_1(x^2)\,\delta^{-1} + \ldots \qquad \ldots(3.17)$$

$$g(x^2) = g_0(x^2) + g_1(x^2)\,\delta^{-1} + \ldots . \qquad \ldots(3.18)$$

Substituting these equations in (3.12) and (3.13) we obtain

$$h_0(x^2) = \phi(x), \; h_1(x^2) = 0$$

$$h_2(x^2) = \frac{2I_0\left(\frac{E}{F} - x^2\right)}{\pi X}\int_a^1 t h_0(t^2)\,dt$$

$$h_3(x) = \frac{2I_3\left(\frac{E}{F} - x^2\right)}{\pi X}\int_a^1 t g_0(t^2)\,dt$$

$$h_4(x^2) = \frac{2}{\pi X}\int_a^1 t\Bigg[\left(\frac{E}{F} - x^2\right)\{h_2(t^2)\,I_0 + g_1(t^2)\,I_3$$
$$+ [I_1 + I_2(t^2 + 3(1 - a^2)/2)]$$
$$\times\, h_0(t^2)\} + I_2\{3x^2(a^2 - x^2) - (E(1 - 2a^2)/F) - a^2\}\,h_0(t^2)\Bigg]\,dt$$

$$h_5(x^2) = \frac{2}{\pi X} \int_a^1 t \Big[\Big(\frac{E}{F} - x^2\Big) \{h_3(t^2)\, I_0 + g_2(t^2)\, I_3$$

$$+ [I_4 + I_5(t^2 + (3(1 - a^2)/2))]$$

$$\times\, g_0(t^2)\} + I_5 \{6x^2(a^2 - x^2) - (E(1 - 2a^2)/F) - a^2\}\, g_0(t^2)\Big]\, dt$$

$$g_0(x^2) = 0,\ g_1(x^2) = 0,\ g_2(x^2) = 0$$

$$g_3(x^2) = \frac{2I_7 x^2 \left(\frac{1 + a^2}{2} - x^2\right)}{\pi X} \int_a^1 t h_0(t^2)\, dt,\ g_4(x^2) = 0$$

$$g_5(x^2) = \frac{2}{\pi X} \int_a^1 t \Big[x^2 \left(\frac{1 + a^2}{2} - x^2\right)(I_8 h_0(t^2) + I_7 h_2(t^2))$$

$$+ I_9 x^2 \left\{(x^2 + t^2)\left(\frac{1 + a^2}{2} - x^2\right) + \tfrac{1}{2}\left(\frac{1 - a^2}{2}\right)^2\right\} h_0(t^2)\Big]\, dt.$$

We now consider the physically important case where the cracks are opened by constant normal pressure $p(x) = p$. Here $h(x^2)$ and $g(x^2)$ are given by

$$h(x^2) = \frac{2p}{\pi X} [(1 + d_0\delta^{-2} + d_1\delta^{-4})(x^2 - E/F)$$

$$+ d_2(x^4 - a^2x^2 + \beta)\delta^{-4} + O(\delta^{-6})] \qquad \ldots(3.19)$$

$$g(x^2) = \frac{2p}{\pi X}\Big[\{e_0\delta^{-3} + (e_1 + e_2x^2)\,\delta^{-5}\}\left(x^2 - \frac{1 + a^2}{2}\right)x^2 + e_3x^2\delta^{-5} + O(\delta^{-6})\Big] \qquad \ldots(3.20)$$

where

$$\beta = \tfrac{1}{3}\{a^2 + (E(1 - 2a^2)/F)\},\ \gamma = -\left(\frac{E}{F} - \frac{a^2 + 1}{2}\right)$$

$$d_0 = I_0\gamma,\ d_1 = d_0^2 + \gamma I_1 + 0.5\, I_2 \left\{\gamma(2 - a^2) + \left(\frac{1 - a^2}{2}\right)^2\right\}$$

$$d_2 = 6 I_2\gamma,\ e_0 = I_7\gamma$$

$$e_1 = \gamma I_8 + \gamma^2 I_0 I_7 + 0.5 I_9 \left\{\gamma(a^2 + 1) + \left(\frac{1 - a^2}{2}\right)^2\right\}$$

$$e_2 = \gamma I_9,\ e_3 = -\,0.5\, e_2 \left(\frac{1 - a^2}{2}\right)^2.$$

The calculations for higher values of δ^{-1} may be carried out similarly but their contributions to stress intensity factors may be negligible.

4. Stress Intensity Factors and Shape of Deformed Cracks

It is of importance to workers in fracture mechanics to estimate the stress intensity factors near crack tips. These intensity factors are defined by the relations

$$K_1 = \lim_{x \to a^-} \sqrt{2(a-x)}\, \sigma_{yy}^-(x, h),\ 0 < x < a \qquad \ldots(4.1)$$

$$K_2 = \lim_{x \to 1^+} \sqrt{2(x-1)}\, \sigma_{yy}^-(x, h),\ x > 1. \qquad \ldots(4.2)$$

Equations (2.8) and (2.11) yield

$$\sigma_{yy}^-(x,h) = -\frac{1}{2\pi}\int_0^\infty \frac{Z_1(\xi, h)\,\psi_1(\xi) + Z_2(\xi, h)\,\psi_2(\xi)}{\sinh^2(2\xi\delta) - (2\xi\delta)^2}\, d\xi \qquad \ldots(4.3)$$

where

$$Z_1(\xi, h) = -\xi\,[(h+\delta)^2\{\cosh 2\xi(h-\delta) - \cosh 2\xi(h+\delta)\} + 8\delta h \sinh \xi(h+\delta)]$$

$$Z_2(\xi, h) = -4\xi^3\delta(h^2-\delta^2) + \xi^2[(h+\delta)^2 \sinh 2\xi(h-\delta)^2 - \sinh 2\xi(h+\delta)] + \xi\,[(h-\delta)\cosh 2\xi(h+\delta) - (h+\delta)\cosh 2\xi(h-\delta) + 2\delta] - 2\sinh \xi(h-\delta)\sinh \xi(h+\delta)\sinh(2\xi\delta).$$

Use of the identity

$$\sinh 2\xi(h-\delta) - \sinh 2\xi(h+\delta) + \sinh(4\xi\delta) = -4\sinh \xi(h-\delta) \times \sinh \xi(h+\delta)\sinh(2\xi\delta)$$

reduces (4.3) in the form

$$\sigma_{yy}^-(x, h) = -\frac{1}{2\pi}\int_0^\infty \xi\psi_2(\xi)\cos \xi x\, d\xi - \frac{1}{2\pi}\int_0^\infty \xi\, \frac{Z_1(\xi, h)\,\psi_1(\xi) + N_{22}(\xi, h)\,\psi_2(\xi)}{\sinh^2 2\xi\delta - (2\xi\delta)^2}\cos \xi x\, d\xi. \qquad \ldots(4.4)$$

The second integral in (4.4) converges uniformly and shall not contribute to stress intensity factors. We shall now evaluate the first integral for $h = 1$. Using eqns. (3.8) and (3.19), we get for $0 < x < a$

$$-\frac{1}{2\pi}\int_0^\infty \xi\psi_2(\xi)\cos\xi x\, d\xi = -\frac{1}{2\pi}\int_a^1 \frac{th(t^2)}{t^2-x^2}\,dt$$

$$= -\frac{p}{2\pi}\Big[(1+d_0\delta^{-2}+d_1\delta^{-4})\Big(1+\frac{x^2-(E/F)}{X}$$

$$+ d_2\Big\{2x^2 - \frac{x^4-a^2x^2+\beta}{X}\Big\}\delta^{-4} + O(\delta^{-6})\Big] \qquad \ldots(4.5)$$

and for $x > 1$

$$-\frac{1}{2\pi}\int_0^\infty \xi\psi_2(\xi)\cos\xi x\, d\xi = -\frac{p}{2\pi}\Big[(1+d_0\delta^{-2}+d_1\delta^{-4})\Big(1+\frac{(E/F)-x^2}{X_1}\Big)$$

$$+ d_2\Big\{\frac{x^4-a^2x^2+\beta}{X_1} - 2x^2\Big\} + O(\delta^{-6})\Big]. \qquad \ldots(4.6)$$

where

$$X_1 = \sqrt{\{(x^2-a^2)(x^2-1)\}}.$$

The stress intensity factors as estimated from (4.1) and (4.2) are

$$K_1 = \frac{-p}{2\pi\sqrt{a(1-a^2)}}[(1+d_0\delta^{-2}+d_1\delta^{-4})(a^2-(E/F))$$
$$+ \beta d_2\delta^{-4} + O(\delta^{-6})] \qquad \ldots(4.7)$$

$$K_2 = \frac{-p}{2\pi\sqrt{(1-a^2)}}[(1+d_0\delta^{-2}+d_1\delta^{-4})((E/F)-1)$$
$$+ d_2(\beta+1-a^2)\delta^{-4} + O(\delta^{-6})]. \qquad \ldots(4.8)$$

The numerical calculations for stress intensity factors K_1 and K_2 are done for $a = 0.1, 0.2, 0.4, 0.6, 0.8$ and for ten values of δ. The variations of stress intensity factors K_1 and K_2 with respect to δ and different values of a are shown in figures 2 and 3 respectively. It is seen that K_1 takes the value for infinite medium when δ approaches 2.5 and K_2 for 2.8. It is further noticed that the variations in stress intensity factors are more pronounced when the crack length is nearly 1 unit; but as the crack length reduces in size the curves become more or less straight line.

We note that as

$$\delta \to \infty,\ K_1 \to p\Big(\frac{E}{F}-a^2\Big)\Big/2\pi\sqrt{a(1-a^2)} \text{ and } K_2 \to p(1-E/F)/2\pi\sqrt{(1-a^2)}$$

in complete agreement with the results for an infinite medium (see Lowengrub and Srivastava 1968).

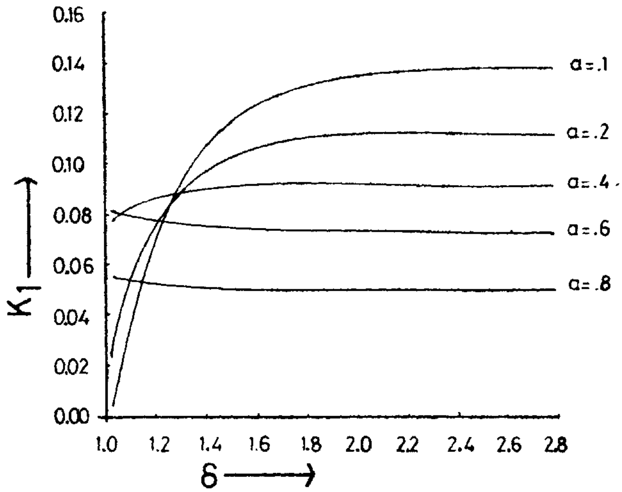

FIG. 2. Variation of K_1 with δ.

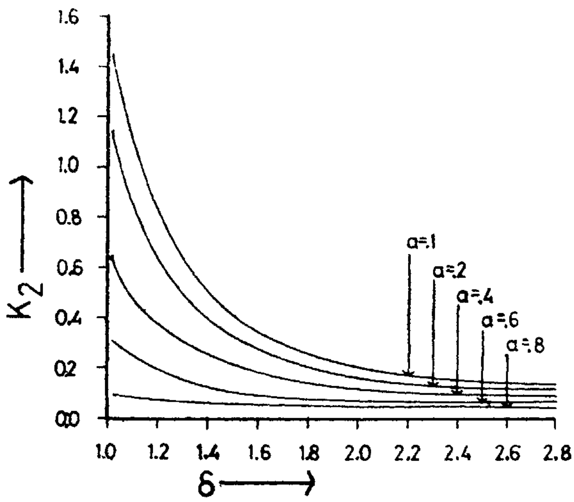

FIG. 3. Variation of K_2 with δ.

We further note that when two cracks merge into a single crack of two units length, then $E \to 1$, $F \to \infty$ and $E/F \to 0$. The constants d's become

$$d_0 = 0.5 I_0,\ d_1 = d_0^2 + 0.5 I_1 + 0.625 I_2,\ d_3 = 3 I_2,\ \gamma = 0.5,\ \beta = 0.$$

The stress intensity factor K_2 takes the form

$$K_2 = \frac{p}{2\pi}\,[1 + 1.1425\delta^{-2} - 14.716\delta^{-4} + O(\delta^{-6})]. \qquad ...(4.9)$$

The expression given by Calhoun and Lowengrub (1974) is

$$K = \frac{p}{2\pi}\,[1 + 1.14\delta^{-2} - 14.268\delta^{-4} - 11.18\delta^{-6} + O(\delta^{-8})].$$

The values of K_2 and K are compared for some values of δ given below:

δ	1.5	2.0	2.5	3.0
K_2	− 1.399	0.366	0.806	0.945
K	− 1.311	0.392	0.816	0.950

We can now obtain an expression for the difference in the displacement $u_y^+(x, h) - u_y^-(x, h)$. The relation (2.6) yields

$$u_y^+(x, h) - u_y^-(x, h) = \sqrt{\frac{\pi}{2}}\,\frac{(1+\eta)}{E}\,\mathcal{F}_c\,[\psi_2(\xi); x].$$

Substituting for $\psi_2(\xi)$ from (3.8), we obtain (for $h = 1$)

$$u_y^+(x, 1) - u_y^-(x, 1) = \frac{\pi(1+\eta)}{2E}\int_x^1 h(t^2)\,dt$$

$$= \frac{p(1+\eta)}{E}\Big[(1 + d_0\delta^{-2} + d_1\delta^{-4})\Big\{E(\phi, k) - \frac{E}{F}\,F(\phi, k)$$

$$d_2\Big\{\frac{2-a^2}{3}\,E(\phi, k) + (\beta - a^2/3)\,F(\phi, k)\Big\}\,\delta^{-4} + O(\delta^{-6})\Big] \quad \ldots(4.10)$$

where $\sin\phi = \sqrt{\{(1-x^2)/(1-a^2)\}}$.

The energy required to open the cracks is given by

$$W = 2p\int_a^1 [u_y^+(x, 1) - u_y^-(x, 1)]\,dx = \frac{\pi(1+\eta)}{E}\int_x^1 th(t^2)\,dt. \quad \ldots(4.11)$$

Substituting for $h(t^2)$ we get

$$W = \frac{-\pi p^2(1+\eta)}{2E}\Big[2\gamma(1 + d_0\delta^{-2} + d_1\delta^{-4})$$

$$- d_2\Big\{2\beta + 1 - \Big(\frac{1+a^2}{2}\Big)^2\Big\}\,\delta^{-4} + O(\delta^{-6})\Big]. \quad \ldots(4.12)$$

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