Indian J. pure appl. Math., **10**(2) : 250-257, February 1979

STRONG SUMMABILITY BY LOGARITHMIC METHOD OF DERIVED FOURIER SERIES AND ITS CONJUGATE SERIES

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(*Received* 24 *April* 1978; *after revision* 6 *July* 1978)

The authors, for the first time, define strong Perron summability and prove two theorems which generalize the theorems of Sharma and Rao (1970), concerning strong (L) summability of derived Fourier series and its conjugate series.

1. INTRODUCTION AND THEOREMS

Let $\Sigma\, a_n$ be an infinite series with the sequence of partial sums $\{S_n\}$. Let $\{p_n\}$ be a sequence of non-negative numbers and suppose that the series $p(x) = \sum_{\nu=1}^{\infty} p_\nu x^\nu$ converges for all x, $0 < x < 1$ and $p(x) \uparrow \infty$ as $x \uparrow 1$. If

$$\sum_{\nu=1}^{\infty} | S_\nu - S |^k p_\nu x^\nu = O(p(x)) \text{ as } x \uparrow 1 \qquad \ldots(1)$$

then the series $\Sigma\, a_n$ is said to be strongly Perron summable with index k or simply summable $[P, k]$. Ordinary and absolute Perron summabilities are discussed by Izumi and Izumi (1970a, b).

For $p_\nu = \frac{1}{\nu}$, $[P, k]$ summability reduces to $[L, k]$ summability (Deokinandan 1968).

Let f be a Lebesgue integrable function, periodic with period 2π, and its Fourier series and conjugate series be

$$\tfrac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) \equiv \sum_{n=0}^{\infty} A_n(t) \qquad \ldots(2)$$

and

$$\sum_{n=1}^{\infty} (b_n \cos nt - a_n \sin nt) \equiv \sum_{n=1}^{\infty} B_n(t) \qquad \ldots(3)$$

respectively.

Then the corresponding derived series are

$$\sum_{n=1}^{\infty} n(b_n \cos nt - a_n \sin nt) \equiv \sum_{n=1}^{\infty} nB_n(t) \qquad \ldots(4)$$

and

$$- \sum_{n=1}^{\infty} n(a_n \cos nt + b_n \sin nt) \equiv - \sum_{n=1}^{\infty} nA_n(t). \qquad ...(5)$$

We shall denote by $\sigma_n(\theta)$ and $\bar{\sigma}_n(\theta)$ the nth partial sums of the series (4) and (5) at $t = \theta$ respectively.

We further write

$$\phi(t) = \tfrac{1}{2}\{f(\theta + t) + f(\theta - t)\}$$

$$\psi(t) = \tfrac{1}{2}\{f(\theta + t) - f(\theta - t)\}$$

$$g(t) \equiv g(\theta, t) = 2\{\psi(t) - tf'(\theta)\}$$

where $f'(\theta)$ denotes the generalized first derivative of $f(t)$ at $t = \theta$,

$$h(t) \equiv h(\theta, t) = 2\{\phi(t) - f(\theta)\}$$

and

$$H(\theta) = \lim H_\lambda(\theta) = \lim \left\{- \frac{1}{4\pi} \int_\lambda^\pi h(t) \operatorname{cosec}^2 \frac{t}{2}\, dt\right\}, \text{ where } \lambda = (1 - x).$$

The object of this paper is to prove the following:

Theorem 1 — If

$$\sum_{\nu=1}^{\infty} \nu p_\nu x^\nu \mid B_\nu(\theta) \mid = o\{(p(x))^2\} \qquad ...(6)$$

$$\int_0^t \mid dg(u) \mid = o\left(\frac{p(1-t)}{p'(1-t)}\right) \text{ as } t \downarrow 0 \qquad ...(7)$$

and

$$\int_t^\pi \frac{\mid dg(u) \mid}{u} = o(p(1-t)) \text{ as } t \downarrow 0 \qquad ...(8)$$

then $$\sum_{\nu=1}^{\infty} \mid \sigma_\nu(\theta) - f'(\theta) \mid p_\nu x^\nu = o\{(p(x))^2\} \text{ as } x \uparrow 1$$

provided that $f'(\theta)$ exists.

Theorem 2 — If $f'(\theta)$ exists and

$$\sum_{\nu=1}^{\infty} \nu p_\nu x^\nu \mid A_\nu(\theta) \mid = o\{(p(x))^2\} \qquad ...(9)$$

$$\int_0^t |\, dh(u)\,| = o\left(\frac{p(1-t)}{p'(1-t)}\right) \text{ as } t \downarrow 0 \qquad \ldots(10)$$

and

$$\int_t^\pi \frac{|\, dh(u)\,|}{u} = o(p(1-t)) \text{ as } t \downarrow 0 \qquad \ldots(11)$$

then $\sum_{\nu=1}^{\infty} |\, \bar{\sigma}_\nu(\theta) - H_\lambda(\theta)\,|\, p_\nu x^\nu = o\{(p(x))^2\}$ as $x \uparrow 1$

where $\lambda = (1-x)$, provided that $H(\theta)$ exists.

It is interesting to note that on taking $p_\nu = 1/\nu$ in our theorems, we obtain two theorems due to Sharma and Rao (1970).

We require the following lemma:

Lemma — (i) $\frac{1}{\pi}\int_0^\pi \cos \nu u\, dg(u) = \nu\, B_\nu(\theta)$,

(ii) $\frac{1}{\pi}\int_0^\pi \sin \nu u\, dh(u) = \nu\, A_\nu(\theta)$.

PROOF OF (i) : We observe that

$$\frac{1}{\pi}\int_0^\pi \cos \nu u\, dg(u) = \frac{2}{\pi}\int_0^\pi \cos \nu u\, d\psi(u)$$

$$= \frac{2\nu}{\pi}\int_0^\pi \sin \nu u\, \psi(u)\, du$$

$$= \frac{\nu}{\pi}\int_{-\pi}^\pi f(u) \sin \nu(u-\theta)\, du$$

$$= \nu(b_\nu \cos \nu\theta - a_\nu \sin \nu\theta)$$

$$= \nu\, B_\nu(\theta).$$

The second part of the Lemma can be proved in a similar manner.

2. Proof of the Theorems

Proof of Theorem 1 — We have

$$\sigma_\nu(\theta) = \frac{1}{2\pi}\int_0^{2\pi}\left\{\frac{d}{d\theta}\,\frac{\sin(\nu+\frac{1}{2})(\theta-u)}{\sin\frac{1}{2}(\theta-u)}\right\} f(u)\,du$$

$$= -\frac{1}{2\pi}\int_0^{2\pi}\left\{\frac{d}{du}\,\frac{\sin(\nu+\frac{1}{2})(\theta-u)}{\sin\frac{1}{2}(\theta-u)}\right\} f(u)\,du$$

$$= -\frac{1}{\pi}\int_0^{\pi}\left\{\frac{d}{du}\,\frac{\sin(\nu+\frac{1}{2})u}{\sin\frac{u}{2}}\right\}\psi(u)\,du$$

$$= \frac{1}{\pi}\int_0^{\pi}\frac{\sin(\nu+\frac{1}{2})u}{\sin\frac{u}{2}}\,d\psi(u)$$

$$= \frac{1}{2\pi}\int_0^{\pi}\frac{\sin(\nu+\frac{1}{2})u}{\sin\frac{u}{2}}\,dg(u) + f'(\theta).$$

Hence

$$\sum_{\nu=1}^{\infty} |\,\sigma_\nu(\theta) - f'(\theta)\,|\,p_\nu x^\nu = \frac{1}{2\pi}\int_0^{\pi}\left\{\sum_{\nu=1}^{\infty}\epsilon_\nu p_\nu x^\nu\,\frac{\sin(\nu+\frac{1}{2})u}{\sin(u/2)}\right\} dg(u)$$

$$= \frac{1}{2\pi}\int_0^{\pi}\left\{\cot\frac{u}{2}\sum_{\nu=1}^{\infty}\epsilon_\nu p_\nu x^\nu \sin\nu u\,dg(u)\right\}$$

$$+ \frac{1}{2\pi}\int_0^{\pi}\left\{\sum_{\nu=1}^{\infty}\epsilon_\nu p_\nu x^\nu \cos\nu u\right\} dg(u)$$

$$= M_1 + M_2, \text{ say}$$

where $\epsilon_\nu = \epsilon_\nu(\theta) = \pm 1$ in such a way so as to make $\epsilon_\nu\{\sigma_\nu(\theta) - f'(\theta)\} \geqslant 0$ for $\nu = 1, 2, 3, \ldots$.

Now, using the first part of the Lemma and (6), we have

$$| M_2 | \leqslant \frac{1}{2\pi} \sum_{\nu=1}^{\infty} p_\nu x^\nu \left| \int_0^\pi \cos \nu u \, dg(u) \right|$$

$$= \tfrac{1}{2} \sum_{\nu=1}^{\infty} \nu p_\nu x^\nu \, | B_\nu(\theta) |$$

$$= o\{(p(x))^2\}.$$

In order to evaluate M_1, we observe that

$$| \omega(u) | = \left| \sum_{\nu=1}^{\infty} \epsilon_\nu p_\nu x^\nu \sin \nu u \right| = O(p(x)) \text{ for } 1 - x \leqslant u \leqslant \pi \qquad \ldots(12)$$

also

$$| \omega(u) | \leqslant x \, | \sin u \, | \, p'(x) \text{ for } 0 \leqslant u \leqslant 1 - x. \qquad \ldots(13)$$

Further, we write

$$M_1 = \frac{1}{2\pi} \left(\int_0^{1-x} + \int_{1-x}^{\pi} \right) \left\{ \cot \frac{u}{2} \, \omega(u) \right\} dg(u)$$

$$= \frac{1}{2\pi} (M_{1,1} + M_{1,2}), \text{ say.}$$

Now, making use of (7) and (13), we have

$$| M_{1,1} | \leqslant \int_0^{1-x} \sin u \, . \, xp'(x) \, . \cot \frac{u}{2} \, | \, dg(u) \, |$$

$$= 2xp'(x) \int_0^{1-x} \cos^2 \frac{u}{2} \, | \, dg(u) \, |$$

$$\leqslant 2xp'(x) \int_0^{1-x} | \, dg(u) \, |$$

$$= o(p(x)).$$

Finally, applying (8) and (12), we obtain

$$M_{1,2} = O(p(x)) \left\{ \int_{1-x}^{\pi} \frac{|dg(u)|}{u} \right\}$$

$$= o\{(p(x))^2\}.$$

This completes the proof of Theorem 1.

Proof of Theorem 2 — We have

$$\bar{\sigma}_\nu(\theta) = -\frac{1}{\pi} \int_{-\pi}^{\pi} f(u) \frac{\partial}{\partial\theta} \left\{ \sum_{r=1}^{\nu} \sin r(u-\theta) \right\} du$$

$$= -\frac{2}{\pi} \int_0^{\pi} \phi(u) \frac{d}{du} \left[\frac{\cos(u/2) - \cos(\nu+\frac{1}{2})u}{2\sin(u/2)} \right] du$$

$$= -\frac{1}{\pi} \int_0^{\pi} \frac{\cos(u/2) - \cos(\nu+\frac{1}{2})u}{2\sin(u/2)} \, dh(u)$$

$$= -\frac{1}{2\pi} \left(\int_0^{1-x} + \int_{1-x}^{\pi} \right) \cot\frac{u}{2} (1 - \cos \nu u) \, dh(u)$$

$$- \frac{1}{2\pi} \int_0^{\pi} \sin \nu u \, dh(u)$$

$$= -\frac{1}{2\pi} (I_1 + I_2 - \frac{1}{2\pi} \int_0^{\pi} \sin \nu u \, dh(u), \text{ say.}$$

Now, using (10), we have

$$|I_1| \leqslant \int_0^{1-x} \left| \cot\frac{u}{2} \,.\, 2\sin^2\frac{\nu u}{2} \right| |dh(u)|$$

$$\leqslant 2\nu \int_0^{1-x} |dh(u)|$$

$$= o\left(\nu \frac{p(x)}{p'(x)}\right).$$

Again,

$$I_2 = \int_{1-x}^{\pi} \cot \frac{u}{2} (1 - \cos \nu u)\, dh(u)$$

$$= \int_{1-x}^{\pi} \cot \frac{u}{2}\, dh(u) - \int_{1-x}^{\pi} \cot \frac{u}{2} \cos \nu u\, dh(u)$$

$$= \left(\cot \frac{u}{2}\, h(u)\right)_{1-x}^{\pi} + \tfrac{1}{2} \int_{1-x}^{\pi} \operatorname{cosec}^2 \frac{u}{2}\, h(u)\, du - \int_{1-x}^{\pi} \cot \frac{u}{2} \cos \nu u\, dh(u).$$

Hence

$$-\frac{1}{2\pi} I_2 = o(1) + H_\lambda(\theta) + \frac{1}{2\pi} \int_{1-x}^{\pi} \cot \frac{u}{2} \cos \nu u\, dh(u).$$

Thus, from the estimates of I_1 and I_2, we have

$$|\bar{\sigma}_\nu(\theta) - H_\lambda(\theta)| = \frac{1}{2\pi} \int_{1-x}^{\pi} \cot \frac{u}{2} \epsilon_\nu \cos \nu u\, dh(u) - \frac{1}{2\pi} \int_0^{\pi} \epsilon_\nu \sin \nu u\, dh(u)$$
$$+ o\left(\nu \frac{p(x)}{p'(x)}\right)$$

where $\epsilon_\nu = \pm 1$ in such a way that $\epsilon_\nu \{\bar{\sigma}_\nu(\theta) - H_\lambda(\theta)\} \geqslant 0$ for $\nu = 1, 2, 3, \ldots$. Hence

$$\sum_{\nu=1}^{\infty} |\bar{\sigma}_\nu(\theta) - H_\lambda(\theta)|\, p_\nu x^\nu = \frac{1}{2\pi} \int_{1-x}^{\pi} \cot \frac{u}{2} \left(\sum_{\nu=1}^{\infty} \epsilon_\nu \cos \nu u\, p_\nu x^\nu \right) dh(u)$$
$$- \frac{1}{2\pi} \int_0^{\pi} \left(\sum_{\nu=1}^{\infty} \epsilon_\nu \sin \nu u\, p_\nu x^\nu \right) dh(u)$$
$$+ o\left(\sum_{\nu=1}^{\infty} \nu p_\nu x^\nu \frac{p(x)}{p'(x)} \right)$$
$$= N_1 + N_2 + o(p(x)), \text{ say.}$$

Using the second part of lemma and (9), we have

$$|N_2| = \left| -\frac{1}{2\pi} \sum_{\nu=1}^{\infty} \int_0^{\pi} \epsilon_\nu \sin \nu u\, p_\nu x^\nu\, dh(u) \right|$$

$$\leqslant \frac{1}{2\pi} \sum_{\nu=1}^{\infty} p_\nu x^\nu \left| \int_0^\pi \sin \nu u \, dh(u) \right|$$

$$= \tfrac{1}{2} \sum_{\nu=1}^{\infty} \nu p_\nu x^\nu \mid A_\nu(\theta) \mid$$

$$= o\{(p(x))^2\}.$$

Finally, since

$$\left| \sum_{\nu=1}^{\infty} \epsilon_\nu \cos \nu u \, p_\nu x^\nu \right| = O(p(x)) \text{ for } 1 - x \leqslant u \leqslant \pi,$$

we have

$$\mid N_1 \mid = \left| \frac{1}{2\pi} \int_{1-x}^{\pi} \cot \frac{u}{2} \left(\sum_{\nu=1}^{\infty} \epsilon_\nu \cos \nu u \, . \, p_\nu x^\nu \right) dh(u) \right|$$

$$= O(p(x)) \, . \left\{ \int_{1-x}^{\pi} \frac{\mid dh(u) \mid}{u} \right\}$$

$$= o\{(p(x))^2\}, \text{ by (11).}$$

This completes the proof of Theorem 2.

ACKNOWLEDGEMENT

The authors are thankful to the referee for his valuable suggestions.

REFERENCES

Deokinandan, B. (1968). On the strong (*L*) summability of Fourier series. *Proc. Am. math. Soc.*, **19**, 33-37.

Izumi, M., and Izumi, S. (1970a). Summability of Fourier series. *Proc. Japan Acad.*, **46**(7), 647-51.

———— (1970b). Absolute summability by logarithmic method of Fourier series. *Proc. Japan Acad.*, **46**(7), 656-59.

Sharma, P. L., and Rao, V. Venu Gopal (1970). On the strong (*L*) summability of the derived Fourier series and its conjugate series. *Studies and Essays presented on Prof. Y. W. Chen's 60th Birthday*, pp. 17-25.