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DECOMPOSITION OF CURVATURE TENSOR FIELD $\underset{2}{K}^{i}_{jkh}$ IN RECURRENT AREAL SPACE

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The purpose of this paper is to decompose the curvature tensor of second kind in a recurrent areal space of first order and to study the properties of such decompositions.

1. INTRODUCTION

A space, in which an m-dimensional area is defined by means of an m-fold integral, is called an Areal space. There are two approaches to introduce the geometrical concept of Areal space. The first approach is due to Davies (1961) and Kawaguchi (1953). In these works a simple m-vector $\pi_{i_1 \ldots i_m}$ satisfying the well-known Plücker relations has been used. The other approach is due to Rund (1966, 1968) in which the use of this m-vector is avoided. This theory is based on a symmetric exploitation of the homogeneity conditions which characterize Lagrangians of parameter invariant integrals. Takano (1967) decomposed the curvature tensor field in a recurrent affinely connected space and studied its properties. Decomposition of curvature tensor fields K^{i}_{jkh} and H^{i}_{jkh} in recurrent Finsler spaces was studied by Pande and Shukla (1977). Pande and Kumar (1977) decomposed Weyl's curvature tensor in recurrent Finsler space and derived certain theorems. The object of this paper is to decompose the curvature tensor of second kind in recurrent Areal space from the point of view of Rund's (1968) approach. After giving the fundamental formulae of Areal space in section 2 we decompose the curvature tensor of second kind in section 3 and study the properties of such decompositions.

Throughout this paper the Latin indices $i, j, k \ldots$ run over 1 to n while Greek indices $\alpha, \beta, \gamma \ldots$ run over 1 to m.

2. FUNDAMENTAL FORMULAE

Let a subspace C_m given by the equation

$$x^i = x^i(t^\alpha) \qquad \ldots(2.1)$$

be immersed in an n-dimensional differentiable manifold X_n, where t^α denotes a system of independent parameters in C_m. Assuming that the functions (2.1) are of class C^4, their first derivatives being denoted by

$$\dot{x}^i_\alpha = \frac{\partial x^i}{\partial t^\alpha} \qquad \text{...(2.2)}$$

we consider a Lagrange function of the form

$$L = L(x^h, \dot{x}^h_s) \qquad \text{...(2.3)}$$

satisfying the conditions :

(a) the Lagrangian L is of class C^4 in all its arguments and it is scalar with respect to the transformation of local coordinates x^i of X_n;

(b) the Lagrangian L is positive for all independent sets of arguments $\dot{x}^i_\alpha$;

(c) the integral

$$I = \int_G L(x^i, \dot{x}^i_\alpha)\, dt^1 \dots dt^m \qquad \text{...(2.4)}$$

over a fixed region G of C_m is invariant under the transformation of the parameters t^α;

(d) The $nm \times nm$ determinant

$$D = \det \left\| \frac{m}{2} \frac{\partial^2 L^{2/m}}{\partial \dot{x}^i_\alpha\, \partial \dot{x}^j_\beta} \right\|$$

is nonvanishing for linearly independent $\dot{x}^i_\alpha$.

The condition (c) is equivalent to the relation (Rund 1966)

$$\frac{\partial L}{\partial \dot{x}^i_\alpha} \dot{x}^i_\beta = \delta^\alpha_\beta L. \qquad \text{...(2.5)}$$

The metric tensor $g^{\alpha\beta}_{ij}(x^h, \dot{x}^h_s)$ defined by (Rund 1968)

$$g^{\alpha\beta}_{ij} = \frac{m}{2} \frac{\partial^2 L^{2/m}}{\partial \dot{x}^i_\alpha\, \partial \dot{x}^j_\beta} \qquad \text{...(2.6)}$$

are the components of a covariant tensor of rank 2 with respect to the transformation of the coordinates of X_n. The relations (2.5) and (2.6) yield

$$L = [m^{-1} g^{\alpha\beta}_{ij}\, \dot{x}^i_\alpha\, \dot{x}^j_\beta]^{m/2} \qquad \text{...(2.7)}$$

from which it is evident that, if L is interpreted as a measure of the area dA of an element of an m-dimensional subspace spanned by $\dot{x}^i_\alpha$ at the point x^j of X_n in the sense that

$$dA = L\, dt^1 \ldots dt^m \qquad \ldots(2.8)$$

then the tensor (2.6) can be regarded as Areal metric tensor and the space equipped with the metric function $L(x^h, \dot{x}^h_\varepsilon)$ as Areal space $A^{(m)}_n$. The covariant derivative (Rund 1968) of a vector field $X^i_\varepsilon(x^h, \dot{x}^h_\varepsilon)$ with respect to x^j is given by

$$X^i_{\varepsilon\,|\,j} = \frac{\partial X^i_\varepsilon}{\partial x^j} - \frac{\partial X^i_\varepsilon}{\partial \dot{x}^l_\lambda} \Gamma^l_{pj}\, \dot{x}^p_\lambda + \Gamma^i_{jl}\, X^l_\varepsilon \qquad \ldots(2.9)$$

where Γ^i_{jl} is a connection coefficient of homogeneous degree zero in its directional arguments.

The commutation formula (Rund 1968) for a vector field $X^i_\varepsilon(x^h, \dot{x}^h_\varepsilon)$ involving the covariant derivative of the type (2.9) is given by

$$X^i_{\varepsilon\,|\,k\,|\,h} - X^i_{\varepsilon\,|\,h\,|\,k} = X^j_\varepsilon \underset{2}{K}{}^i_{jkh} \frac{\partial X^i_\varepsilon}{\partial \dot{x}^l_\alpha} \underset{1}{K}{}^l_{pkh}\, \dot{x}^p_\alpha + X^i_{\varepsilon\,|\,j}\, T^j_{kh} \qquad \ldots(2.10)$$

where

$$\underset{1}{K}{}^l_{pkh} = -\underset{1}{K}{}^l_{phk} \qquad \ldots(2.11)$$

$$\underset{2}{K}{}^i_{jkh} = -\underset{2}{K}{}^i_{jhk} \qquad \ldots(2.12)$$

and

$$T^i_{kh} = \Gamma^i_{kh} - \Gamma^i_{hk} \qquad \ldots(2.13)$$

are curvature tensor of first kind, curvature tensor of second kind and torsion tensor respectively.

The curvature tensor of second kind (Rund 1968) satisfy the identity

$$\underset{2}{K}{}^i_{jkh} + T^i_{jk\,|\,h} + T^i_{lj}\, T^l_{kh} + \text{cycl}\,(j, k, h) = 0 \qquad \ldots(2.14)$$

where cycl (j, k, h) denotes the sum of two sets of three terms each obtained by replacing the indices (j, k, h) by first (k, h, j) and then by (h, j, k) in the expression on the left-hand side of (2.14).

The Bianchi identities (Rund 1968) corresponding to the curvature tensor of first and second kinds are given by

$$\underset{2}{K}{}^{i}_{jkh\,|\,l} + \frac{\partial \Gamma^{i}_{kj}}{\partial \dot{x}^{p}_{\alpha}} \underset{1}{K}{}^{p}_{qhl}\, \dot{x}^{q}_{\alpha} + T^{q}_{kh}\, \underset{2}{K}{}^{i}_{jql} + \text{cycl}\,(k, l, h) = 0 \qquad \ldots(2.15)$$

where cycl (k, l, h) has the same meaning as in (2.14).

Definition 2.1 — Areal space $A^{(m)}_{n}$ is said to be $\underset{2}{K}$-recurrent Areal space of first order, if the curvature tensor $\underset{2}{K}{}^{i}_{jkh}$ satisfies the relation

$$\underset{2}{K}{}^{i}_{jkh\,|\,l} = v_l\, \underset{2}{K}{}^{i}_{jkh} \qquad \ldots(2.16)$$

where v_l is a non-zero vector field.

3. Decomposition of $\underset{2}{K}{}^{i}_{jkh}$ in a Recurrent Areal Space

Let us consider the decomposition of $\underset{2}{K}{}^{i}_{jkh}\,(x^h, \dot{x}^h_{\varepsilon})$ as follows :

$$\underset{2}{K}{}^{i}_{jkh} = X^{i}_{j}\, \phi_{kh} \qquad \ldots(3.1)$$

and

$$X^{i}_{j}\, v_i = p_j \qquad \ldots(3.2)$$

where X^{i}_{j}, ϕ_{kh} are non-zero homogeneous tensor fields of degree zero and v_i, p_j are the recurrent vector and decomposed vector fields respectively.

We have the following theorems :

Theorem 3.1 — In a $\underset{2}{K}$-recurrent Areal space, if X^{i}_{j} is covariantly invariant then the tensor field ϕ_{kh} satisfies the recurrency condition i.e.

$$\phi_{kh\,|\,l} = v_l \phi_{kh}. \qquad \ldots(3.3)$$

Proof : Differentiating (3.1) covariantly with respect to x^l and using the relation (2.16), we get

$$v_l\, \underset{2}{K}{}^{i}_{jkh} = X^{i}_{j\,|\,l}\, \phi_{kh} + X^{i}_{j}\, \phi_{kh\,|\,l}. \qquad \ldots(3.4)$$

In view of (3.1), the relation (3.4) gives

$$\nu_l X^i_j \phi_{kh} = X^i_{j|l} \phi_{kh} + X^i_j \phi_{kh|l}. \qquad ...(3.5)$$

Since X^i_j is covariantly invariant (i.e. $X^i_{j|l} = 0$), then eqn. (3.5) reduces to

$$X^i_j (\phi_{kh|l} - \nu_l \phi_{kh}) = 0. \qquad ...(3.6)$$

As X^i_j is a non-zero tensor therefore the relation (3.6) yields the Theorem 3.1.

Theorem 3.2 — Under the decomposition (3.1), the decomposition tensor field ϕ_{kh} satisfies the identity :

$$\phi_{kh} + \phi_{hk} = 0. \qquad ...(3.7)$$

PROOF : In the light of the relation (3.1), (2.12) can be written as

$$X^i_j (\phi_{kh} + \phi_{hk}) = 0. \qquad ...(3.8)$$

Since X^i_j is non-zero so (3.8) yields the required identity (3.7).

Theorem 3.3 — In a $\underset{2}{K}$-recurrent Areal space, the identity for the torsion tensor is given by

$$p_{[j} \phi^*_{kh]} = - \nu_i (T^i_{[jk|h]} + T^i_{l[j} T^l_{kh]}), \qquad ...(3.9)$$

under the decomposition (3.1).

PROOF : Using eqn. (3.1) in (2.14), we get

$$X^i_{[j} \phi_{kh]} = - (T^i_{[jk|h]} + T^i_{l[j} T^l_{kh]}). \qquad ...(3.10)$$

Transvecting (3.10) by ν_i and taking into consideration eqn. (3.2), we obtain Theorem 3.3.

Theorem 3.4 — In a $\underset{2}{K}$-recurrent Areal space, the following identity holds under the decomposition (3.1).

$$\nu_{[l} \phi_{kh]} + T^q_{[kh} \phi^{**}_{<q>l]} = 0. \qquad ...(3.11)$$

$^*A_{[jkh]} = A_{jkh} + A_{khj} + A_{hjk}$.

**The indices in brackets < > are free from symmetric and skew-symmetric parts.

PROOF : In view of the relations (2.16) and (3.1), eqn. (2.15) takes the form

$$v_l X^i_j \phi_{kh} + v_h X^i_j \phi_{lk} + v_k X^i_j \phi_{hl} + \frac{\partial \Gamma^i_{kj}}{\partial \dot{x}^p_\alpha} \underset{1}{K}^p_{qhl} \dot{x}^q_\alpha + \frac{\partial \Gamma^i_{lj}}{\partial \dot{x}^p_\alpha} \underset{1}{K}^p_{qkh} \dot{x}^q_\alpha$$

$$+ \frac{\partial \Gamma^i_{hj}}{\partial \dot{x}^p_\alpha} \underset{1}{K}^p_{qlk} \dot{x}^q_\alpha + T^q_{kh} X^i_j \phi_{ql} + T^q_{lk} X^i_j \phi_{qh} + T^q_{hl} X^i_j \phi_{qk} = 0. \quad \text{...(3.12)}$$

Transvecting (3.12) by v_i and using (3.2) and (2.11), we obtain

$$p_j [v_l \phi_{kh} + v_h \phi_{lk} + v_k \phi_{hl} + T^q_{kh} \phi_{ql} + T^q_{lk} \phi_{qh} + T^q_{hl} \phi_{qk}]$$

$$= v_i \dot{x}^q_\alpha \left(\frac{\partial \Gamma^i_{kj}}{\partial \dot{x}^p_\alpha} \underset{1}{K}^p_{qlh} + \frac{\partial \Gamma^i_{lj}}{\partial \dot{x}^p_\alpha} \underset{1}{K}^p_{qhk} + \frac{\partial \Gamma^i_{hj}}{\partial \dot{x}^p_\alpha} \underset{1}{K}^p_{qkl} \right). \quad \text{...(3.13)}$$

Multiplying (3.13) by $\dot{x}^p_\alpha$ and taking into consideration the homogeneity property of the connection coefficient, we have

$$p_j \dot{x}^p_\alpha (v_l \phi_{kh} + v_h \phi_{lk} + v_k \phi_{hl} + T^q_{kh} \phi_{ql} + T^q_{lk} \phi_{qh} + T^q_{hl} \phi_{qk}) = 0. \quad \text{...(3.14)}$$

Since p_j and $\dot{x}^p_\alpha$ are not zero, then we get (3.11) from (3.14).

Theorem 3.5 — In a $\underset{2}{K}$-recurrent Areal space, under the decomposition (3.1), the relation

$$(v^{***}_{[l \mid m]} - v_s T^s_{lm}) \phi_{kh} = (\phi_{hs} X^s_k + \phi_{sk} X^s_h) \phi_{lm} \quad \text{...(3.15)}$$

holds.

PROOF : Differentiating (3.2) covariantly with respect to x^m and using (3.3), we have

$$\phi_{kh \mid l \mid m} = v_{l \mid m} \phi_{kh} + v_l v_m \phi_{kh}. \quad \text{...(3.16)}$$

Interchanging the indices l, m in (3.16) and subtracting the result thus obtained from (3.16) and using the commutation formula (2.10) first then simplifying the result thus obtained in the light of (3.1), (3.2) and (3.7), we have

***$v_{[l \mid m]} = v_{l \mid m} - v_{m \mid l}$

$$v_{[l \mid m]}\phi_{kh} + \phi_{sh} X^s_k \phi_{lm} + \phi_{ks} X^s_h \phi_{lm} + v_s \phi_{hk} T^s_{lm} = \frac{\partial \phi_{kh}}{\partial \dot{x}^p_\alpha} \underset{1}{K}{}^p_{qml} \dot{x}^q_\alpha . \quad \ldots(3.17)$$

Now multiplying (3.17) by $\dot{x}^p_\alpha$ and taking into account the homogeneity property of ϕ_{kh}, we get

$$\dot{x}^p_\alpha \left(v_{[l \mid m]}\phi_{kh} + \phi_{sh} X^s_k \phi_{lm} + \phi_{ks} X^s_h \phi_{lm} + v_s \phi_{hk} T^s_{lm}\right) = 0. \quad \ldots(3.18)$$

Since $\dot{x}^p_\alpha$ is not zero so using eqn. (3.7) in (3.18), we obtain Theorem 3.5.

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