Indian J. pure appl. Math., **10**(3) : 271-276, March 1979

BLOCKWISE LEE-BURST-ERROR-CORRECTING CODES

R. S. AGGARWAL*

Department of Mathematics, University of Delhi, Delhi 110007

(*Received* 27 *April* 1978; *after revision* 30 *November* 1978)

The paper contains a lower and an upper bound on the number of parity checks for a linear code (over the field Z_q of integers modulo a prime integer q), which is capable of correcting phased Lee-burst errors with specified lengths and weights.

1. INTRODUCTION

Blockwise Lee-burst-error-correcting Codes

In most of the communication channels disturbances like lightning, break down and loose connections affect successive digits for some length of the word, such errors are known as 'clustered errors' or 'burst errors'.

Let us consider an (n, k) block code C over the field $Z_q = \{0, 1, 2, ..., q - 1\}$ of integers modulo and odd prime integer q. A vector $(a_1, a_2, ..., a_n) \in C$ is called a Lee-burst of length b and weight w if all the non-zero entries in this vector are confined to b consecutive positions, the first and last of which are non-zero and $W_L(a_1, a_2, ..., a_n) = w$ where $W_L(a_1, a_2, ..., a_n)$ denotes the Lee weight of $(a_1, a_2, ..., a_n)$ defined by

$$W_L(a_1, a_2, ..., a_n) = \sum_{i=1}^{n} | a_i |$$

where
$$| a_i | = \begin{cases} a_i & \text{if } 0 \leqslant a_i \leqslant \frac{1}{2}q, \\ -a_i & (\text{mod } q) \text{ if } \frac{1}{2}q < a_i \leqslant q - 1. \end{cases}$$

We know that for a linear code capable of correcting bursts of length b or less, no code vector should be expressible as a sum of two bursts of length b or less each (refer Peterson 1961). As a consequence, a burst of length greater than or equal to $n/2$ cannot be corrected by the conventional burst error correcting codes of word length n. Similarly, burst of weight more than $n(q - 1)/4$ cannot be corrected by these codes.

The above type of errors may be corrected by considering a code formed of different sub-blocks of different lengths such that the pattern of possible errors in

*This work is supported by a Teacher Fellowship awarded to the author by the University Grants Commission, New Delhi, India.

each sub-block is known. Such errors are termed as 'phased burst errors'. Codes capable of correcting such errors in Hamming case were discussed by Sharma and Gupta (1976). As discussed by Berlekemp (1968), in phase modulation schemes Lee weight is more appropriate than Hamming weight.

Our aim in this paper is to examine the structure and capabilities of codes correcting Lee burst errors spread or phased over different parts of the word length.

In our studies we extensively use the combinatorial results concerning the filling up of certain number of positions by integers from $\{0, 1, ..., (q-1)/2\}$, to make up a specified sum [refer Berge (1971) and Goel (1976) for details].

It may be remarked here that the field elements a_i and $q - a_i$ are equivalent for the purpose of determining the weight and therefore each non-zero entry in a vector is less than or equal to $(q-1)/2$ with two equivalent repetitive values.

2. Lower Bound on Parity Checks

Theorem 2.1 — An (n, k) linear block code (over the field Z_q of integers modulo an odd prime q), which is capable of correcting phased Lee-burst of length $b_1, b_2, ..., b_m$ or less with weights $w_1, w_2, ..., w_m$ or less from the first, second, ..., mth part of each code word divided into blocks of lengths $n_1, n_2, ..., n_m$ respectively such that $\sum_{s=1}^{m} n_s = n$ should have a lower bound on parity checks given by

$$n - k \geqslant \sum_{s=1}^{m} \log_q [(1 + 2n_s \,.\, \min((q-1)/2, w_s)$$

$$+ \sum_{r_s=2}^{b_s} \sum_{N_s=2}^{w_s} \sum_{\delta_{s_2}=1}^{\min(N_s-\delta_{s_1}, (q-1)/2)} \sum_{\delta_{s_1}=1}^{\min(N_s-1, (q-1)/2)}$$

$$\times \sum_{k_s=\{(N_s-\delta_{s_1}-\delta_{s_2})/(r_s-2)\}}^{\min(N_s-\delta_{s_1}-\delta_{s_2}, (q-1)/2)} (n_s - r_s + 1)\, A^{k_s}(N_s - \delta_{s_1} - \delta_{s_2}, r_s - 2)\, 2^2] \quad ...(2.1)$$

where $A^{k_s}(N_s - \delta_{s_1} - \delta_{s_2}, r_s - 2)$* denotes the number of arrangements of filling $r_s - 2$ positions with integers from $\{0, 1, 2, ..., (q-1)/2\}$ with each non-zero entry

---

$$*A^{k_s}(N_s - \delta_{s_1} - \delta_{s_2}, r_s - 2) = \sum_{t_0, t_1, ..., t_{k_s}} \frac{(r_s - 2)\,!}{t_0\,!\, t_1\,! \,...\, t_{k_s}!}\, 2^{t_1 + ... + t_{k_s}}$$

where t_i denotes the number of times the integer i occurs and therefore t_i's are non-negative integers such that

$$\sum_{i=0}^{k_s} t_i = r_s - 2,\ t_k \geqslant 1 \text{ and } \sum_{i=1}^{k_s} i \,.\, t_i = N_s - \delta_{s_1} - \delta_{s_2}.$$

having two equivalent repetitive values to make up the sum $N_s - \delta_{s_1} - \delta_{s_2}$ and having the largest entry K_s.

PROOF : We count the number of correctable error patterns which in the sth block of length n_s are Lee bursts of length b_s or less with weight w_s or less.

It is clear that there is only one burst of length zero, namely the null vector. The number of bursts of length 1 with weight w_s or less in a vector of length n_s is $2n_s \,.\, \min(w_s, (q-1)/2)$. Now the number of bursts of length say r_s $(2 \leqslant r_s \leqslant b_s)$ with weight say N_s is

$$(n_s - r_s + 1)\, A^{k_s}(N_s - \delta_{s_1} - \delta_{s_2}, r_s - 2)\, 2^2$$

where δ_{s_1} is an arbitrary value assigned to one of the end points non-zero entry and for a fixed δ_{s_1}, δ_{s_2} is the value assigned to the other end point non-zero entry, clearly, $1 \leqslant \delta_{s_1} \leqslant \min(N_s - 1, (q-1)/2)$ and $1 \leqslant \delta_{s_2} \leqslant \min(N_s - \delta_{s_1}, (q-1)/2)$. Here k_s denotes the largest entry in each arrangement. Clearly,

$$\left\{\frac{N_s - \delta_{s_1} - \delta_{s_2}}{r_s - 2}\right\} \leqslant K_s \leqslant \min(N_s - \delta_{s_1} - \delta_{s_2}, (q-1)/2).$$

Allowing δ_{s_1}, δ_{s_2}, and k_s to vary, the total number of bursts of length b_s or less with weight w_s or less in such vectors is given by B

$$B_L(n_s, b_s, w_s) = 1 + 2n_s \,.\, \min((q-1)/2, w_s) + \sum_{r_s}\sum_{N_s}\sum_{\delta_{s_2}}\sum_{\delta_{s_1}}\sum_{k_s} (n_s - r_s + 1)\, A^{k_s}(N_s - \delta_{s_1} - \delta_{s_2}, r_s - 2)\, 2^2. \quad \text{...(2.2)}$$

Now the total number of all correctable Lee bursts is $\prod_{s=1}^{m} B_L(n_s, b_s, w_s)$. Since there should be at least as many cosets as is the number of correctable error patterns, so

$$q^{n-k} \geqslant \prod_{s=1}^{m} B_L(n_s, b_s, w_s). \quad \text{...(2.3)}$$

Taking logarithms to the base q and using (2.2) the required condition (2.1) is obtained.

Deductions

I. Setting $n_1 = n_2 = \ldots = n_{m-1} = 0$, $n_m = n$, $b_m = b$ and $w_m = w$ in (2.1) we get the result for a code correcting bursts of length b or less with weight w or less due to Sharma and Goel (1978).

II. Setting $q = 3$ also in I, we get a result due to Sharma and Dass (1974).

III. Putting $q = 3$ in (2.1), we get the result for blockwise Hamming burst error correcting codes, which coincides with the result obtained by Sharma and Gupta (1976) for $q = 3$.

3. Upper Bound on Parity Checks

It is quite possible that for a code discussed in section 2, the lower bound is satisfied for certain parameters but for such parameters, the code may not exist. In this section we derive a sufficient condition for the existence of such codes. The following Lemma is useful for our purpose.

Lemma 3.1 — The total number of n-vectors which can be expressed as sum of two bursts of length b or less and Lee weight w or less is given by

$$S_L(n, b, w) = B_L(n, b, 2w) + \sum_{i=1}^{b-1} \left\{ n - (b + i - 1) \right\} \cdot B_i + \sum_{i=0}^{n-2b} \left\{ n - (2b + i - 1) \right\} \cdot C \qquad \ldots(3.1)$$

where

$$B_i = \sum_{r_1, r_2, r_3} \left[\sum_{\delta_1=1}^{\min((q-1)/2, r_1)} \; \sum_{k_1=\{(r_1-\delta_1)/(i-1)\}}^{\min(r_1-\delta_1-i+2, (q-1)/2)} A^{k_1}(r_1 - \delta_1, i - 1)\, 2 \right]$$
$$\times \left[\sum_{\delta_2=1}^{\min((q-1)/2, r_2)} \; \sum_{k_2=\{(r_2-\delta_2)/(i-1)\}}^{\min(r_2-\delta_2-i+2, (q-1)/2)} A^{k_2}(r_2 - \delta_2, i - 1)\, 2 \right]$$
$$\times \left[\sum_{k_3=\{r_3/(b-i)\}}^{\min(r_3-b+i+1, (q-1)/2)} A^{k_3}(r_3, b - i) \right] \text{for } i > 1,$$

$$\text{and } B_1 = 4 \sum_{k_3} A^{k_3}(r_3, b - 1) \qquad \ldots(3.2)$$

$$r_1 \leqslant w,\ r_2 \leqslant w,\ r_1 + r_2 + r_3 \leqslant 2w$$

and

$$C = \left[\sum_{r_1=1}^{w} \; \sum_{\delta_1=1}^{\min((q-1)/2, r_1)} \; \sum_{k_1=\{(r_1-\delta_1)/(b-1)\}}^{\min(r_1-\delta_1-b+2, (q-1)/2)} A^{k_1}(r_1 - \delta_1, b - 1)2 \right] \times$$

(equation continued on p. 275)

$$\times\left[\sum_{r_2=1}^{w}\ \sum_{\delta_2=1}^{\min((q-1)/2,\, r_2)}\ \sum_{k_2=\{(r_2-\delta_2)/(b-1)\}}^{\min(r_2-\delta_2-b+2,\,(q-1)/2)} A^{k_2}(r_2-\delta_2,\, b-1)\, 2\right]. \quad \ldots(3.3)$$

Theorem 3.2 — A sufficient condition for the existence of a code discussed in Theorem (2.1) is given by

$$q^{n-k} \geqslant \left[\prod_{s=1}^{m-1} S_L(n_s, b_s, w_s)\right]\cdot J_L(n_m, b_m, w_m) \quad \ldots(3.4)$$

where $S_L(n, b, w)$ is given by (3.1) and $J_L(n, b, w)$ is given by

$J_L(n, b, w) =$

$$\left[1+\sum_{\theta=2}^{w}\ \sum_{\delta=1}^{\min(\theta-1,\,(q-1)/2)}\ \sum_{k=\{(\theta-\delta)/(b-1)\}}^{\min(\theta-\delta-b+2,\,(q-1)/2)} A^{k}(\theta-\delta,\, b-1)\, 2\right] B_L(n-b, b, w)$$

$$+\left[\sum_{\theta=w+1}^{2w}\ \sum_{\delta=1}^{\min(\theta-1,\,(q-1)/2)}\ \sum_{k=\{(\theta-\delta)/(b-1)\}}^{\min(\theta-\delta-b+2,\,(q-1)/2)} A^{k}(\theta-\delta,\, b-1)\, 2\right]$$

$$+\sum_{p=1}^{b-1}\ \sum_{p_1, p_2, p_2}\left[\left\{\binom{b-p-1}{p_1}\sum_{\delta=1}^{w-2}\ \sum_{k=1}^{\min(\delta,\,(q-1)/2)} A^{k}(\delta, p_1)\right\}\right.$$

$$\times\left\{\binom{p}{p_2}\sum_{\delta=1}^{2w-2}\ \sum_{k=1}^{\min(\delta,\,(q-1)/2} A^{k}(\delta, p_2)\right\}$$

$$\times\left\{\binom{b-p-1}{p_3-1}\sum_{\delta=2}^{w}\ \sum_{\delta_1=1}^{\min(\delta-1,\,(q-1)/2)}\right.$$

$$\left.\left.\times\sum_{k=1}^{\min(\delta-1,\,(q-1)/2)} A^{k}(\delta-\delta_1, p_3-1)\, 2\right\}\right]$$

where restrictions on p_i's and their largest weight contributions are as given below:

Restrictions on p_i's	*Largest weight contributions*
$0 \leqslant p_1 \leqslant b-2$	$w-2$
$1 \leqslant p_2 \leqslant b-1$	$2w-2$
$1 \leqslant p_3 \leqslant b$	w
$p_1+p_2 \leqslant b-1$	$w-1$
$p_1+p_2+p_3 \leqslant 2b-1$	$2w-1$

PROOF : The method followed is the one given by Sacks (1958) for the well-known Varsharmov-Gilbert bound. The condition may be obtained by examining the possibility of constructing the corresponding parity check matrix by proceeding in the manner used by Sacks.

Deductions

I. Putting $n_1 = n_2 = \ldots = n_{m-1} = 0$, $n_m = n$, $b_m = b$ and $w_m = w$ in (3.4) the condition of blockwise correction is dispensed away with a sufficient condition for the existence of an (n, k) code capable of correcting bursts of length b or less with weight w or less due to Sharma and Goel (1978).

II. Setting $q = 3$ also in I, we get the result due to Dass (1975).

III. Putting $q = 3$ in (3.4), a sufficient condition for existence of an (n, k) code capable of correcting Hamming phased burst errors is obtained. This result coincides with the result obtained by Sharma and Gupta (1976) for $q = 3$.

ACKNOWLEDGEMENT

The author wishes to thank Dr B. D. Sharma for helpful suggestions and illuminating discussions.

REFERENCES

Berge, C. (1971). Principles of Combinatorics. Academic Press, Inc., New York.

Berlekemp, E. R. (1968). Algebraic Coding Theory. McGraw-Hill Book Co., Inc., New York.

Dass, B. K. (1975). A sufficient bound for codes correcting bursts with weight constraint. *J. Assoc. Comput. Mach.*, **22**, 501-503.

Goel, S. N. (1976). A study in error correcting codes through Lee distance. Ph.D. Thesis, Meerut University, Meerut.

Peterson, W. W. (1961). Error Correcting Codes. The MIT Press, Cambridge, Mass.

Sacks, G. E. (1958). Multiple error correction by means of parity checks. *IRE Trans.*, IT-4, 145-47.

Sharma, B. D., and Dass, B. K. (1974). Extended Varsharmov-Gilbert and sphere packing bounds for burst correcting codes. *IEEE Trans.*, IT-20, 291-92.

Sharma, B. D., and Goel, S. N. (1978). Bounds over codes correcting bursts having a given Lee-weight or intensity. *Rev. Roum. Math. Pures Appl.*, Tome XXIII, No. 6.

Sharma, B. D., and Gupta, S. N. (1976). Unequal burst error correcting codes, Chapter III. Ph.D. Thesis, University of Delhi, Delhi.