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TOTIENTS AND UNITARY TOTIENTS WITH RESPECT TO A SET OF POLYNOMIALS

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For any given integer coefficient polynomials $f_1 = f_1(x), f_2 = f_2(x), \ldots, f_s = f_s(x)$ of positive degrees and positive integers $u_1, u_2, \ldots, u_s$ and k, and multiplicative function $\eta(n)$, the totients functions $\Phi_F^{k,\eta}(n)$, $\phi_F^{k,\eta}(n)$ and their unitary analogues $\Phi_{F*}^{k,\eta}(n)$ and $\phi_{F*}^{k,\eta}(n)$ are defined and studied where F is written, for simplicity, for $f_1, u_1; f_2, u_2; \ldots; f_s, u_s$. When $s = 1$, for special choices of $k, f_1(x)$, u_1 and $\eta(n)$, $\Phi_F^{k,\eta}(n)$ includes as special cases, Euler's totient function and its various generalizations and $\Phi_{F*}^{k,\eta}(n)$ includes the unitary analogues of some of these functions studied earlier. The well-known identities $\sum_{d \mid n} \phi(d) = n$ and $\phi(ab)\,\phi((a, b)) = \phi(a)\,\phi(b)\,(a, b)$ are generalized for these functions and the average orders of these functions are obtained subject to certain restrictions on the sizes of $N_{f_i}(n)$ and $\eta(n)$, $N_f(n)$ being the number of incongruent solutions mod n of $f(x) \equiv O \pmod{n}$.

§1. *Introduction and preliminaries* — Let $V = V_t$ be the set of all ordered t-tuples of integers $X = \{x_i\}_{i=1}^t$ called integral t-vectors or simply t-vectors. Two t-vectors $X = \{x_i\}_{i=1}^t$ and $Y = \{y_i\}_{i=1}^t$ are called congruent modulo the positive integer n, denoted $X \equiv Y \pmod{n}$, if $x_i \equiv y_i \pmod{n}$ for $1 \leqslant i \leqslant t$. Clearly there are n^t t-vectors which are incongruent mod n and we call any such set of n^t t-vectors which are incongruent mod n, a complete residue system of t-vectors mod n.

Let $f_1 = f_1(x), f_2 = f_2(x), \ldots, f_s = f_s(x)$ be any integer coefficient polynomials of positive degrees, $u_1, u_2, \ldots, u_s$ positive integers

$$t = u_1 + u_2 + \ldots + u_s \text{ and } \mathcal{F} = F_1 F_2 \ldots F_t \qquad \ldots(1.1)$$

be any permutation of $f_1, f_2, \ldots, f_s$ where for each i, f_i appears u_i times. Let us say a t-vector $X = \{x_i\}_{i=1}^t$ is k-prime to n with respect to $\mathcal{F}$ if $((F_1(x_1), F_2(x_2), \ldots, F_t(x_t)), n)_k = 1$, where the symbol $(a, b, c, \ldots, e)_k$ stands for the largest kth power common divisor of $a, b, c, \ldots, e$ with the convention $(0, 0, \ldots, 0)_k = 0$ and

$(a, b, c, \ldots, e)_1 = (a, b, c, \ldots, e)$. The number of t-vectors incongruent mod n which are k-prime to n with respect to $\mathcal{F}$ will turn out to be depending only on the f_i and their frequencies u_i but not on the permutation $\mathcal{F}$ and this number we denote by

$$\Phi^{(k)}_{f_1, u_1; f_2, u_2; \ldots; f_s, u_s}(n) \text{ or simply } \Phi^{(k)}_F(n) \qquad \ldots(1.2)$$

writing F for $f_1, u_1; f_2, u_2; \ldots; f_s, u_s$.

Clearly, when $s = 1$, this function reduces to the totient function $\Phi^{(k)}_{f_1, u_1}(n)$ studied by Chidambaraswamy (1974)† which itself includes as special cases Euler's totient function and its various generalizations.

In fact,

$$\left.\begin{array}{ll} \text{(i)}\ \Phi^{(1)}_{x,1}(n) = \phi(n), & \text{(ii)}\ \Phi^{(1)}_{x,t}(n) = J_t(n), \\ \text{(iii)}\ \Phi^{(k)}_{x,1}(n) = \Phi_k(n), & \text{(iv)}\ \Phi^{(k)}_{x,1}(n^k) = \phi_k(n), \\ \text{(v)}\ \Phi^{(1)}_{f,1}(n) = \phi_f(n) & \end{array}\right\} \qquad \ldots(1.3)$$

where $\phi(n)$, $J_t(n)$, $\Phi_k(n)$, $\phi_k(n)$ and $\phi_f(n)$ are respectively the totient functions of Euler, Jordan Dickson (1952), Klee (1948), Cohen (1956) and Menon (1967). $\Phi^{(2)}_{x,1}(n)$ was also studied by Haviland (1944).

We also have the so-called unitary analogues of some of these functions in the literature; for example $\phi^*(n)$ (Cohen 1960), $\phi^*_k(n)$ and $J^*_k(n)$ (Rao 1966). Let us call a t-vector $X = \{x_i\}_{i=1}^{t}$ k-unitarily k-prime to n with respect to $\mathcal{F}$ if $((F_1(x_1), F_2(x_2), \ldots, F_t(x_t)), n)^*_k = 1$ where $(a, n)^*_k$ stands for the largest kth power common divisor of a and n which is a k-unitary divisor of n. We recall (Chidambaraswamy 1970) a divisor d of n is called a k-unitary divisor of n if $\left(d, \frac{n}{d}\right)_k = 1$. Here again, the number of distinct t-vectors mod n which are k-unitarily k-prime to n with respect to $\mathcal{F}$ does not depend on the order in which the f_i's appear in $\mathcal{F}$ and this number we denote by

$$\Phi^{(k)*}_{f_1, u_1; f_2, u_2; \ldots f_s, u_s}(n) \text{ or simply by } \Phi^{(k)}_{F^*}(n). \qquad \ldots(1.4)$$

When $s = 1$, $f_1(x) = x$, we have,

$$\Phi^{(1)}_{x^*,1}(n) = \phi^*(n),\ \Phi^{(1)}_{x^*,t}(n) = J^*_t(n),\ \Phi^{(k)}_{x^*,1}(n^k) = \phi^*_k(n). \qquad \ldots(1.5)$$

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†In Chidambaraswamy (1974), the symbol $\phi^k_f(t; n)$ was used for $\Phi^{(k)}_{f,t}(n)$.

In fact, these functions can further be generalized a little. Specifically, if $\eta = \eta(n)$ is any multiplicative function, i.e. $\eta(mn) = \eta(m)\,\eta(n)$ whenever $(m, n) = 1$, we shall define the functions $\Phi_F^{k,\eta}(n)$ and $\Phi_{F*}^{k,\eta}(n)$ which when $\eta(n) = 1$ for all n, reduce respectively to $\Phi_F^{(k)}(n)$ and $\Phi_{F*}^{(k)}(n)$. Also when $s = 1$, $f_1(x) = x$, $u_1 = 1$, $k = 1$, and for special choice of $\eta(n)$, $\Phi_F^{k,\eta}(n)$ reduces to the generalized Euler totient function introduced recently by Venkataraman and Sivaramakrishnan (1972).

In this paper we study these more general functions $\Phi_F^{k,\eta}(n)$ and $\Phi_{F*}^{k,\eta}(n)$ and their restrictions to the kth powers of integers. These are all multiplicative and satisfy identities generalizing that of Gauss:

$$\sum_{d \mid n} \phi(d) = n \qquad \ldots(1.6)$$

and like $\Phi_{f,t}^{(k)}(n)$ (Chidambaraswamy 1974), $\Phi_F^{k,\eta}(n)$ satisfies an identity which is a generalization of the well-known

$$\phi(ab)\,\phi((a, b)) = \phi(a)\,\phi(b)\,(a, b). \qquad \ldots(1.7)$$

We also obtain the average order of these functions subject to certain restrictions on the size of $N_{f_i}(n)$ and (or) that of $\eta(n)$ where $N_f(n)$ stands for the number of incongruent solutions mod n of

$$f(x) \equiv O \,(\text{mod } n). \qquad \ldots(1.8)$$

Since $N_f(n)$ is a multiplicative function of n, so is

$$Q_F(n) = N_{f_1}^{u_1}(n)\, N_{f_2}^{u_2}(n) \ldots N_{f_s}^{u_s}(n) \qquad \ldots(1.9)$$

where

$$N_{f_i}^{u_i}(n) = (N_{f_i}(n))^{u_i}.$$

Let, for any multiplicative function $\eta(n)$,

$$\left.\begin{array}{ll} \text{(i)}\ \ \mu_F^{(k)}(n) = \mu(n)\, Q_F(n^k) & \text{(ii)}\ \ \mu_{F*}^{k}(n) = \mu^*(n)\, Q_F(n^k) \\ \text{(iii)}\ \ \mu_F^{k,\eta}(n) = \eta(n)\, \mu_F^{(k)}(n) & \text{(iv)}\ \ \mu_{F*}^{k,\eta}(n) = \eta(n)\, \mu_{F*}^{(k)}(n) \end{array}\right\} \qquad \ldots(1.10)$$

where $\mu(n)$ is the Mobius function and $\mu^*(n)$ is the unitary analogue of the Mobius function (*see* Cohen 1960). We recall $\mu^*(n) = (-1)^{\omega(n)}$, $\omega(n)$ being the number of distinct prime factors of n. Clearly all the functions in (1.10) are multiplicative and

$$\mu_F^{k,l}(n) = \mu_F^{(k)}(n),\ \mu_{F*}^{k,l}(n) = \mu_{F*}^{(k)}(n) \qquad \text{...(1.11)}$$

if $\quad I(n) = 1$ for $n \geqslant 1$.

We shall also need

$$\left.\begin{aligned} \nu_F^{(k)}(n) &= 1 \text{ if } n = 1 \\ &= 0 \text{ if } n \text{ is not a } k\text{th power} \\ &= \prod_{p \mid n} Q_F^{\alpha}(p^k), \text{ if } n \text{ is} \end{aligned}\right\} \qquad \text{...(1.12)}$$

a kth power, and $p^{\alpha k}$ is the highest power of p dividing n;

$$M_F^{(k)}(n) = \nu_F^{(k)}(n^k);\ M_F^{k,n}(n) = \eta(n)\, M_F^{(k)}(n); \qquad \text{...(1.13)}$$

$$\left.\begin{aligned} \nu_{F*}^{(k)}(n) &= 1 \text{ if } n = 1 \\ &= 0 \text{ if } n \text{ is not a } k\text{th power} \\ &= \prod_{p \mid n} Q_F(p^{\alpha k}), \text{ if } n \text{ is a } k\text{th} \end{aligned}\right\} \qquad \text{...(1.14)}$$

power and $p^{\alpha k}$ is the highest power of p dividing n;

$$M_{F*}^{(k)}(n) = \nu_{F*}^{(k)}(n^k);\ M_{F*}^{k,n}(n) = \eta(n)\, M_{F*}^{(k)}(n). \qquad \text{...(1.15)}$$

Let $\mu_u(n)$ be the generalized mobius function defined by Venkataraman and Sivaramakrishnan (1972), i.e.

$$\left.\begin{aligned} \mu_u(u) &= 1 \text{ if } n = 1 \\ &= 0 \text{ if } n \text{ is not square free} \\ &= e(\omega(n), 2u), \text{ if } n \text{ is square free} \end{aligned}\right\} \qquad \text{...(1.16)}$$

where

$$e(a, b) = e^{2\pi a i/b},$$

and

$$\eta_u(n) = \mu(n)\, \mu_u(n). \qquad \text{...(1.17)}$$

Clearly $\nu_F^{(k)}(n)$, $\nu_{F*}^{k}(n)$, $M_F^{(k)}(n)$, $M_F^{k,n}(n)$, $M_{F*}^{(k)}(n)$, $M_{F*}^{k,n}(n)$, $\mu_u(n)$ and $\eta_u(n)$ are all multiplicative. We recall that a divisor d of n is called a unitary divisor of n if $n = d\delta$, $(d, \delta) = 1$ and we use $d \parallel n$ to mean that d is a unitary divisor of n. We shall use the trivial estimates,

$$\sum_{n \leqslant x} n^t = \frac{x^{t+1}}{t+1} + O(x^t),\ x \geqslant 1,\ t \geqslant 0 \qquad \text{...(1.18)}$$

$$\sum_{n \leqslant x} \frac{1}{n^r} = O(x^{1-r}),\ 0 < r < 1,\ x \geqslant 1 \qquad \ldots(1.19)$$

$$\sum_{n > x} \frac{1}{n^r} = O(x^{1-r}),\ r > 1,\ x \geqslant 1 \qquad \ldots(1.20)$$

in finding the average orders.

§2. Let us denote by $\Phi_{\mathcal{F}}^{(k)}(n)$ the number of distinct t-vectors mod n which are k-prime to n with respect to $\mathcal{F}$. We shall obtain a formula for $\Phi_{\mathcal{F}}^{(k)}(n)$, which formula shows its independence of the order of the f_i's in $\mathcal{F}$.

Denoting $(F_1(x_1), F_2(x_2), \ldots, F_t(x_t))$ by $\langle X \rangle_{\mathcal{F}}$, it is easily seen that, if

$$X = \{x_i\}_{i=1}^t \text{ and } Y = \{y_i\}_{i=1}^t$$

$$X \equiv Y (\text{mod } n) \text{ implies } (\langle X \rangle_{\mathcal{F}}, n)_k = (\langle Y_i \rangle_{\mathcal{F}}, n)_k \qquad \ldots(2.1)$$

and $$(m, n) = 1 \text{ implies } (\langle X \rangle_{\mathcal{F}}, mn)_k = (\langle X \rangle_{\mathcal{F}}, m)_k\, (\langle X \rangle_{\mathcal{F}}, n)_k. \qquad \ldots(2.2)$$

Let $(m, n) = 1$; for every pair of t-vectors $A = \{a_i\}_{i=1}^t$ (mod m) and $B = \{b_i\}_{i=1}^t$ mod n, there corresponds uniquely, by Chinese reminder theorem, a vector $C = \{c_i\}_{i=1}^t$ (mod mn) such that

$$C \equiv A \ (\text{mod } m),\ C \equiv B \ (\text{mod } n) \qquad \ldots(2.3)$$

and conversely. Also from (2.2), (2.3) and (2.1)

$$(\langle C \rangle_{\mathcal{F}}, mn)_k = (\langle A \rangle_{\mathcal{F}}, m)_k\, (\langle B \rangle_{\mathcal{F}}, n)_k \qquad \ldots(2.4)$$

which shows that

$$\Phi_{\mathcal{F}}^{(k)}(n) \text{ is a multiplicative function of } n. \qquad \ldots(2.5)$$

We now determine $\Phi_{\mathcal{F}}^{(k)}(p^\alpha)$, p a prime and $\alpha \geqslant 0$.

Case 1 — $\alpha < k$; in this case every t-vector mod p^α is k-prime to p^α with respect to $\mathcal{F}$.

Case 2 — $\alpha \geqslant k$; let $\alpha = \lambda k + \theta$, $0 \leqslant \theta < k$; a t-vector $X = \{x_i\}_{i=1}^t$ is not k-prime to p^α with respect to $\mathcal{F}$ if and only if

$$F_i(x_i) \equiv O(\bmod p^k),\ 1 \leqslant i \leqslant t; \qquad \ldots(2.6)$$

i.e. if and only if u_i components of X are solutions of

$$f_i(x) \equiv O(\bmod p^k),\ 1 \leqslant i \leqslant s.$$

Since in a complete residue system of integers mod p^α, there are $p^{\alpha-k}$ complete residue systems mod p^k, the number of distinct t-vectors mod p^α which are not k-prime to p^α with respect to $\mathcal{F}$ is

$$\{p^{\alpha-k}N_{f_1}(p^k)\}^{u_1}\{p^{\alpha-k}N_{f_2}(p^k)\}^{u_2}\ldots\{p^{\alpha-k}N_{f_s}(p^k)\}^{u_s}$$

$$= p^{(\alpha-k)t}\,Q_F(p^k).$$

Since $Q_F(p^k)$ does not depend on the order of the f_i in the permutation $\mathcal{F}$, we see $\Phi^{(k)}_{\mathcal{F}}(p^\alpha)$ and hence by (2.5) $\Phi^{(k)}_{\mathcal{F}}(n)$ does not depend on the order of the f_i in the permutation. Hence we have the following

Theorem 2.1 — (a) $\Phi^{(k)}_F(n)$ is a multiplicative function of n.

(b) $$\Phi^{(k)}_F(p^\alpha) = p^{\alpha t}, \quad \text{if } \alpha < k$$

$$= p^{\alpha t}\left\{1 - \frac{Q_F(p^k)}{p^{Kt}}\right\}, \text{ if } \alpha \geqslant k.$$

We have, from Theorem 2.1, and (i) of (1.10),

$$\Phi^{(k)}_F(n) = n^t \prod_{p^k \mid n}\left\{1 - \frac{Q_F(p^k)}{p^{kt}}\right\}$$

$$= n^t \sum_{d^k \mid n} \frac{\mu(d)\,Q_F(d^k)}{d^{kt}} = \sum_{d^k \mid n} \mu^{(k)}_F(d)\left(\frac{n}{d^k}\right)^t. \qquad \ldots(2.7)$$

We define, for any multiplicative function $\eta(n)$,

$$\Phi^{k,\eta}_F(n) = \sum_{d^k \mid n} \mu^{k,\eta}_F(d)\left(\frac{n}{d^k}\right)^t = n^t \prod_{p^k \mid n}\left\{1 - \frac{Q_F(p^k)\,\eta(p)}{p^{kt}}\right\} \qquad \ldots(2.8)$$

and

$$\Phi^{k,\eta}_F(n) = \Phi^{k,\eta}_F(n^k),\ \phi^{(k)}_F(n) = \Phi^{(k)}_F(n^k). \qquad \ldots(2.9)$$

Clearly,

$$\Phi^{k,I}_F(n) = \Phi^{(k)}_F(n),\ \phi^{k,I}_F(n) = \phi^{(k)}_F(n) \qquad \ldots(2.10)$$

and when $s = 1$, $f_1(x) = x$, $u_1 = 1$, $k = 1$, $\eta(n) = \eta_u(n)$

$$\Phi_F^{k,\eta} u(n) = \phi_{\mu_u}(n) \qquad ...(2.11)$$

where $\phi_{\mu_u}(n)$ is the generalized Euler's function defined by Venkataraman and Sivaramakrishnan (1972), with a slight change of notation.

All these functions are multiplicative, and

$$\phi_F^{(k)}(n) = n^{kt} \prod_{p \mid n} \left\{1 - \frac{Q_F(p^k)}{p^{kt}}\right\} = \sum_{d \mid n} \mu_F^{(k)}(d) \left(\frac{n}{d}\right)^{kt} \qquad ...(2.12)$$

$$\phi_F^{k,\eta}(n) = n^{kt} \prod_{p \mid n} \left\{1 - \frac{Q_F(p^k)\,\eta(p)}{p^{kt}}\right\} = \sum_{d \mid n} \mu_F^{k,\eta}(d) \left(\frac{n}{d}\right)^{kt}. \qquad ...(2.13)$$

The proofs of Corollaries 2 and 3 of Chidambaraswamy (1974) can be adjusted to give the proofs of Theorems 2.2 and 2.3 below.

Theorem 2.2 — If $\eta(n)$ is completely multiplicative, i.e. $\eta(mn) = \eta(m)\,\eta(n)$ for all positive integers m and n.

$$\sum_{d^k \mid n} \Phi_F^{k,\eta}\left(\frac{n}{d^k}\right) M_F^{k,\eta}(d) = n^t.$$

From Theorem 2.2, (2.10), (1.12) and (1.13), we have the following:

Corollary 2.2.1 — (1) $\sum_{d \mid n} \phi_F^{k,\eta}(d)\, M_F^{k,\eta}\left(\frac{n}{d}\right) = n^{kt}$, if $\eta(n)$ is completely multiplicative,

(2) $$\sum_{d^k \mid n} \Phi_F^{(k)}\left(\frac{n}{d^k}\right) \nu_F^{(k)}(d^k) = n^t$$

(3) $$\sum_{d \mid n} \phi_F^{(k)}(d)\, M_F^{(k)}\left(\frac{n}{d}\right) = n^{kt}.$$

Theorem 2.3 — If $m = m_1^k m_2$, $n = n_1^k n_2$ where m_2 and n_2 are k-free, i.e. not divisible by the kth power of any prime,

$$\Phi_F^{k,\eta}(mn)\, \Phi_F^{k,\eta}((m, n)_k) \left\{\frac{m_2 n_2}{(m_1^k n_1^k, m_2 n_2)_k}\right\}^t$$

$$= \Phi_F^{k,\eta}(m)\, \Phi_F^{k,\eta}(n)\, \Phi_F^{k,\eta}\left(\frac{m_2 n_2}{(m_1^k n_1^k, m_2 n_2)_k}\right) (m, n)_k^t$$

Corollary 2.3.1 — (1) $\Phi_F^{(k)}(mn)\,\Phi_F((m, n)_k)\left\{\dfrac{m_2n_2}{(m_1^k\, n_1^k\,, m_2n_2)_k}\right\}^t$

$$= \Phi_F^{(k)}(m)\,\Phi_F^{(k)}(n)\,\Phi_F^{(k)}\left(\frac{m_2n_2}{(m_1^k\, n_1^k\,, m_2n_2)_k}\right)(m, n)_k^t,$$

if $m = m_1^k\, m_2$, $n = n_1^k\, n_2$, where m_2 and n_2 are k-free.

(2) $\phi_F^{k,\eta}(mn)\,\phi_F^{k,\eta}((m, n)) = \phi_F^{k,\eta}(m)\,\phi_F^{k,\eta}(n)\,(m, n)^{kt}$

(3) $\phi_F^{(k)}(mn)\,\phi_F^{(k)}((m, n)) = \phi_F^{(k)}(m)\,\phi_F^{(k)}(n)\,(m, n)^{kt}$.

From (2) of Corollary 2.3.1, we have

$$\phi_F^{k,\eta}(m)\,\phi_F^{k,\eta}(n)\,(m, n)^{kt} = \phi_F^{k,\eta}(mn)\,\phi_F^{k,\eta}((m, n))$$

$$= \phi_F^{k,\eta}((m, n)\,[m, n])\,\phi_F^{k,\eta}\Big(((m, n), [m, n])\Big)$$

$$= \phi_F^{k,\eta}(m, n)\,\phi_F^{k,\eta}([m, n])\,(m, n)^{kt}$$

where $[m, n]$ is the l.c.m. of m and n and hence we have

$$\phi_F^{k,\eta}(m)\,\phi_F^{k,\eta}(n) = \phi_F^{k,\eta}(m, n)\,\phi_F^{k,\eta}([m, n]) \qquad \ldots(2.14)$$

and from which we have

$$\phi_F^{(k)}(m)\,\phi_F^{(k)}(n) = \phi_F^{(k)}((m, n))\,\phi_F^{(k)}([m, n]). \qquad \ldots(2.15)$$

Clearly, (2.8) and (2.13) show that $\Phi_F^{k,\eta}(n)$ and $\phi_F^{k,\eta}(n)$ are $\neq 0$, for all n, if for all primes p, $Q_F(p^k)\,\eta(p) \neq p^{kt}$.

Likewise, $\Phi_F^{(k)}(n)$ and $\phi_F^{(k)}(n)$ are $\neq 0$ if for all primes p, $Q_F(p^k) \neq p^{kt}$. In fact, a proof similar to that of Theorem 3 of Chidambaraswamy (1974) gives the following:

Theorem 2.4 — If for some i in $1 \leqslant i \leqslant s$, $N_{f_i}(p) < p$ for all primes p, then $\Phi_F^{(k)}(n)$ and $\phi_F^{(k)}(n)$ are $\neq 0$ for all n and they tend to ∞ as n tends to infinity.

We shall now prove the following:

Theorem 2.5 — If either

$$N_{f_i}(n) = O(n^{\epsilon}),\ 0 < \epsilon < 1, \text{ for some } i \text{ in } 1 \leqslant i \leqslant s \qquad \text{...(2.16)}$$

and

$$\eta(n) = o(n^{\delta}),\ 0 < \delta < (k-1) + ku_i(1-\epsilon) \qquad \text{...(2.17)}$$

hold or

$$\eta(n) = O(n^{\delta}),\ k - \delta > 1, \text{ holds} \qquad \text{...(2.18)}$$

the average order of $\Phi_F^{k,\eta}(n)$ is αn^t, where

$$\alpha = \sum_{n=1}^{\infty} \frac{\mu_F^{k,\eta}(n)}{n^{k(t+1)}}. \qquad \text{...(2.19)}$$

PROOF : We observe that (2.16) and (1.9) imply

$$Q_F(n) = O(n^{u_1+u_2+\cdots+u_i\epsilon+\cdots+u_r}) = O(n^{t-u_i(1-\epsilon)}). \qquad \text{...(2.20)}$$

Now, it is easily seen that if (2.16) and (2.17) hold,

$$\frac{\mu_F^{k,\eta}(n)}{n^{k(t+1)}} = O\left(\frac{1}{n^{k+ku_i(1-\epsilon)-\delta}}\right)$$

while if (2.18) holds, it is $O\left(\frac{1}{n^{k-\delta}}\right)$ and so in either case, the series in (2.19) converges absolutely.

We have from (2.8)

$$\sum_{n\leqslant x} \Phi_F^{k,\eta}(n) = \sum_{d^k\delta\leqslant x} \delta^t \mu_F^{k,\eta}(d)$$

$$= \sum_{d^k\leqslant x} \mu_F^{k,\eta}(d) \sum_{\delta\leqslant x/d^k} \delta^t$$

$$= \sum_{d^k\leqslant x} \mu_F^{k,\eta}(d) \left\{\frac{\left(\frac{x}{d^k}\right)^{t+1}}{t+1} + O\left(\frac{x}{d^k}\right)^t\right\}. \qquad \text{...(2.21)}$$

Suppose (2.16) and (2.17) hold. Then the r.h.s. of (2.21) is $\alpha \frac{x^{t+1}}{t+1} + E_1 + E_2$, where

$$E_1 = O\left(x^{t+1} \sum_{d > x^{1/k}} \frac{1}{d^{k+ku_i(1-\epsilon)-\delta}}\right) = O(x^{t+(1/k)+(\delta/k)-u_i(i-\epsilon)})$$

and

$$E_2 = O\Big(x^t \sum_{d \leqslant x^{1/k}} \frac{1}{d^{ku_i(1-\epsilon)-\delta}}\Big)$$

$$= O(x^t), \text{ if } ku_i(1-\epsilon) - \delta > 1;$$

$$= O(x^t \log x), \text{ if } ku_i(1-\epsilon) - \delta = 1;$$

$$= O(x^{t+(1/k)+(\delta/k)-u_i(1-\epsilon)}), \text{ if } 0 \neq ku_i(1-\epsilon) - \delta < 1;$$

$$= O(x^{t+(1/k)}), \text{ if } ku_i(1-\epsilon) - \delta = 0.$$

Now, Since $\frac{1}{k} + \frac{\delta}{k} - u_i(1-\epsilon) < 1$ and since $\delta = ku_i(1-\epsilon)$ implies $k > 1$, the result is clear in this case.

Suppose (2.18) holds. In this case, the r.h.s of (2.21) is

$$\alpha \cdot \frac{x^{t+1}}{t+1} + O\Big(x^{t+1} \sum_{d > x^{1/k}} \frac{1}{d^{k-\delta}}\Big) + O\Big(x^t \sum_{d < x^{1/k}} d^\delta\Big).$$

Since $\sum_{d > x^{1/k}} \frac{1}{d^{k-\delta}} = O(x^{-1+(1+\delta)/k})$, and

$$\sum_{d < x^{1/k}} d^\delta = O(1), \text{ if } \delta < -1;$$

$$= O(x^t \log x), \text{ if } \delta = -1;$$

$$= O(x^{(1+\delta)/k}), \text{ if } -1 < \delta;$$

and since $1 + \delta < k$, the result is clear in this case also and the proof of the theorem is complete.

Since for $I(n)$ (2.17) is always satisfied and (2.18) holds for $k > 1$, we have the following.

Corollary 2.5.1 — If either $k > 1$ or (2.16) holds, the average order of $\Phi_F^{(k)}(n)$ is βn^t, where

$$\beta = \sum_{n=1}^{\infty} \frac{\mu_F^{(k)}(n)}{n^{k(t+1)}}.$$

A proof similar to that of Theorem 2.5 gives

Theorem 2.6 — If, either (2.16) and

$$\eta(n) = O(n^{\delta}),\ 0 < \delta < ku_i(1 - \epsilon) \qquad \ldots(2.22)$$

hold, or

$$\eta(n) = O(n^{\delta}),\ \delta < 0 \qquad \ldots(2.23)$$

holds, the average order of $\phi_F^{k,\eta}(n)$ is γn^{kt}, where

$$\gamma = \sum_{n=1}^{\infty} \frac{\mu_F^{k,\eta}(n)}{n^{kt+1}}.$$

From Theorem 2.6, we have the following:

Corollary 2.6.1 — If (2.16) holds the average order of $\phi_F^{(k)}(n)$ is δn^{kt}, where

$$\delta = \sum_{n=1}^{\infty} \frac{\mu_F^{(k)}(n)}{n^{kt+1}}.$$

§3. We now take up the unitary analogues of the functions studied in §2.

Let us denote by $\Phi_{\mathcal{F}^*}^{(k)}(n)$ the number of t-vectors which are k-unitarily k-prime to n with respect to $\mathcal{F}$ of (1.1). It is easily verified that (2.1), (2.2), (2.3), and (2.4) hold if in them we replace $(a, b)_k$ by $(a, b)_k^*$ and we get

$$\Phi_{\mathcal{F}^*}^{(k)}(n) \text{ is a multiplicative function of } n. \qquad \ldots(3.1)$$

Let $n = p^{\alpha}$, p a prime and $\alpha \geqslant 0$. If $\alpha \geqslant k$, and $\alpha = \lambda k + \theta$, $0 \leqslant \theta < k$, the only kth power divisors of p^{α} which are k-unitary divisors of p^{α} are 1 and $p^{\lambda k}$ and we see that a t-vector $X = \{x_i\}_{i=1}^{t}$ is not k-unitarily k-prime to p^{α} with respect to $\mathcal{F}$ if and only if

$$F_i(x_i) \equiv O(p^{\lambda k}),\ i = 1, 2, \ldots, t;$$

i.e. if and only if u_i components of X are solutions of

$$f_i(x) \equiv O(\text{mod } p^{\lambda k}),\ i = 1, 2, \ldots, s. \qquad \ldots(3.2)$$

Thus, the number of t-vectors mod p^{α} which are not k-unitarily k-prime to p^{α} with respect to $\mathcal{F}$ is

$$= \{p^{\theta} N_{f_1}(p^{\lambda k})\}^{u_1} \{p^{\theta} N_{f_2}(p^{\lambda k})\}^{u_2} \{p^{\theta} N_{f_s}(p^{\lambda k})\}^{u_s}$$

$$= p^{\theta t} Q_F(p^{\lambda k}),$$

if $\alpha < k$, clearly every t-vector mod p^{α} is k-unitarily k-prime to p^{α} with respect to $\mathcal{F}$ and we have the following:

Theorem 3.1 — (a) $\Phi_{F^*}^{(k)}(n)$ is a multiplicative function of n.

(b) $\Phi_{F^*}^{(k)}(p^{\alpha}) = p^{\alpha t}$, if $\alpha < k$

$$= p^{\alpha t}\left\{1 - \frac{Q_F(p^{\lambda k})}{p^{\lambda k t}}\right\},$$

if $\lambda k \leqslant \alpha < (\lambda + 1)k$.

We have from Theorem 3.1,

$$\Phi_{F^*}^{(k)}(n) = n^t \prod_{\substack{p^{\lambda k} \mid n \\ p^{(\lambda+1)k} \mid n}} \left\{1 - \frac{Q_F(p^{\lambda k})}{p^{\lambda k t}}\right\} \qquad \text{...(3.3)}$$

and this is easily shown to be

$$= \sum_{\substack{d^k \mid n \\ (d^{k},n \mid d^k)_k = 1}} \mu_{F^*}^{(k)}(d)\left(\frac{n}{d^k}\right)^t. \qquad \text{...(3.4)}$$

We define, for any multiplicative function $\eta(n)$,

$$\Phi_{F^*}^{k,\eta}(n) = \sum_{\substack{d^k \mid n \\ (d^k, n/d^k)_k = 1}} \mu_{F^*}^{k,\eta}(d)\left(\frac{n}{d^k}\right)^t$$

$$= n^t \prod_{\substack{p^{\lambda k} \mid n \\ p^{(\lambda+1)k} \mid n}} \left\{1 - \frac{Q_F(p^{\lambda k})\,\eta(p^{\lambda})}{p^{\lambda k t}}\right\} \qquad \text{...(3.5)}$$

$$\phi_{F^*}^{k,\eta}(n) = \Phi_{F^*}^{k,\eta}(n^k), \quad \phi_{F^*}^{(k)}(n) = \Phi_{F^*}^{(k)}(n^k). \qquad \text{...(3.6)}$$

All these functions are multiplicative and we have

$$\phi_{F^*}^{k,\eta}(n) = \sum_{d \| n} \mu_{F^*}^{k,\eta}(d)\left(\frac{n}{d}\right)^{kt} = n^{kt} \prod_{p^{\alpha} \| n} \left\{1 - \frac{Q_F(p^{\alpha k})\,\eta(p^{\alpha})}{p^{\alpha k t}}\right\}$$

...(3.7)

$$\phi_{F*}^{(k)}(n) = \sum_{d \| n} \mu_{F*}^{(k)}(d) \left(\frac{n}{d}\right)^{kt} = n^{kt} \prod_{p^{\alpha} \| n} \left\{1 - \frac{Q_F(p^{\alpha k})}{p^{\alpha kt}}\right\}. \qquad \ldots(3.8)$$

Theorem 3.2 —
$$\sum_{\substack{d^k \mid n \\ (d^k, n/d^k)_k = 1}} \Phi_{F*}^{k,\eta}\left(\frac{n}{d^k}\right) M_{F*}^{k,\eta}(d) = n^t. \qquad \ldots(3.9)$$

PROOF : The r.h.s. of (3.9) is multiplicative and it is easily verified that the l.h.s. of it is also so, using the fact that if $(m, n) = 1$ every $d^k \mid mn$ can be uniquely expressed as $d^k = d_1^k d_2^k$, $d_1^k \mid m$ and $d_2^k \mid n$ and that $(a, b) = (m, n) = (a, n) = (b, m) = 1$ implies $(ab, mn)_k = (a, m)_k (b, n)_k$. Hence it is enough if we verify (3.9) for $n = p^{\alpha}$, p a prime and $\alpha \geqslant 0$. If $\alpha < k$, 1 is the only kth power divisor of p^{α} which is k-unitary divisor of p^{α} and so in this case, the l.h.s. of (3.9) is $\Phi_{F*}^{k,\eta}(p^{\alpha}) = p^{\alpha t}$, while if $\alpha \geqslant k$, $\alpha = \lambda k + \theta$, $0 \leqslant \theta < k$, the only kth power divisors of p^{α} which are k-unitary divisors of p^{α} are 1 and $p^{\lambda k}$ and so the l.h.s. of (3.9) in this case, is

$$\Phi_{F*}^{k,\eta}(p^{\alpha}) M_{F*}^{k,\eta}(1) + \Phi_{F*}^{k,\eta}(p^{\theta}) M_{F*}^{k,\eta}(p^{\lambda})$$

$$= p^{\alpha t}\left\{1 - \frac{Q_F(p^{\lambda k})\,\eta(p^{\lambda})}{p^{\lambda k t}}\right\} + p^{\theta t} Q_F(p^{\lambda k})\,\eta(p^{\lambda})$$

$= p^{\alpha t}$ and the theorem follows.

We have from Theorem 3.2, (3.6), (1.14) and (1.15), the following.

Corollary 3.2.1 — (1)
$$\sum_{\substack{d^k \mid n \\ (d^k, n/dk)_k = 1}} \Phi_{F*}^{(k)}\left(\frac{n}{d^k}\right) \nu_{F*}^{(k)}(d^k) = n^t$$

(2)
$$\sum_{d \| n} \phi_{F*}^{k,\eta}(d) M_{F*}^{k,\eta}\left(\frac{n}{d}\right) = n^{kt}$$

(3)
$$\sum_{d \| n} \phi_{F*}^{(k)}(d) M_{F*}^{(k)}\left(\frac{n}{d}\right) = n^{kt}.$$

Clearly, (3.5) and (3.7) show that $\Phi_{F*}^{k,\eta}(n)$ is $\neq 0$ if for all primes p such that $p^k \mid n$, $p^{\lambda k t} \neq Q_F(p^{\lambda k})\,\eta(p^{\lambda})$, where $p^{\lambda k} \mid n$, and $p^{(\lambda+1)k} \mid n$ and that $\phi_{F*}^{k,\eta}(n) \neq 0$ if for all $p^{\alpha} \| n$, $p^{\alpha k t} \neq Q_F(p^{\alpha k})\,\eta(p^{\alpha})$. A proof similar to that of Theorem 3 of Chidambaraswamy (1974) gives

Theorem 3.3 — If for some i in $1 \leqslant i \leqslant r$, $N_{f_i}(p) < p$ for all primes p, then $\Phi_{F^*}^{(k)}(n)$ and $\phi_{F^*}^{(k)}(n)$ are $\neq 0$ for all n and tend to ∞ as $n \to \infty$.

Lemma — $$\sum_{\substack{m \leqslant x \\ (m,n)_k = 1}} m^t = \frac{x^{t+1}}{t+1} \sum_{d^k \mid n} \frac{\mu(d)}{d^k} + O(x^t 2^{\omega_k(n)}),$$

where $\omega_k(n)$ is the number distinct primes whose kth powers divide n.

PROOF : $$\sum_{\substack{m < x \\ (m,n)_k = 1}} m^t = \sum_{m \leqslant x} m^t \sum_{d^k \mid (m,n)_k} \mu(d) = \sum_{d^k \mid n} d^{kt} \mu(d) \sum_{\delta \leqslant x/d^k} \delta^t$$

$$= \sum_{d^k \mid n} d^{kt} \mu(d) \left\{ \frac{(x/d^k)^{t+1}}{t+1} + O\left(\frac{x}{d^k}\right)^t \right\}$$

$$= \frac{x^{t+1}}{t+1} \sum_{d^k \mid n} \frac{\mu(d)}{d^k} + O\left(x^t \sum_{d^k \mid n} \mid \mu(d) \mid \right)$$

$$= \frac{x^{t+1}}{t+1} \sum_{d^k \mid n} \frac{\mu(d)}{d^k} + O(x^t 2^{\omega_k(n)}).$$

We have, from the lemma,

$$\sum_{\substack{m \leqslant x \\ (m, n^k)_k = 1}} m^t = \frac{x^{t+1}}{t+1} \frac{\phi_k(n)}{n^k} + O(x^t 2^{\omega(n)}). \qquad \ldots(3.10)$$

We shall now prove the following:

Theorem 3.4 — If either (2.16) and (2.17) hold or (2.18) holds, the average order of $\Phi_{F^*}^{k,\eta}(n)$ is $\alpha^* n^t$, where

$$\alpha^* = \sum_{n=1}^{\infty} \frac{\mu_{F^*}^{k,\eta}(n) \phi_k(n)}{n^{k(t+2)}} \qquad \ldots(3.11)$$

PROOF : It is easily seen, as in the proof of Theorem 2.5, that (3.11) converges absolutely under the hypothesis of the theorem. We have from (3.5),

$$\sum_{n \leqslant x} \Phi_{F^*}^{k,\eta}(n) = \sum_{n \leqslant x} \sum_{\substack{n = d^k\delta \\ (d^k, \delta)_k = 1}} \mu_{F^*}^{k,\eta}(d)\, \delta^t$$

$$= \sum_{d^k \leqslant n} \mu_{F^*}^{k,\eta}(d) \sum_{\substack{\delta \leqslant x/d^k \\ (\delta, d^k)_k = 1}} \delta^t, \text{ which by (3.10) is}$$

$$= \sum_{d^k \leqslant x} \mu_{F^*}^{k,\eta}(d) \left\{ \frac{\left(\frac{x}{d^k}\right)^{t+1}}{t+1} \frac{\phi_k(d)}{d^k} + O\left(\left(\frac{x}{d^k}\right)^t 2^{\omega(d)}\right) \right\}$$

$$= \frac{x^{t+1}}{t+1} \sum_{d^k \leqslant x} \frac{\mu_{F^*}^{k,\eta}(d)\, \phi_k(d)}{d^{k(t+2)}} + O\left(x^t \sum_{d^k \leqslant x} \frac{|\mu_{F^*}^{k,\eta}(d)|\, 2^{\omega(d)}}{d^{kt}}\right). \quad \ldots(3.12)$$

Now, we have

$$2^{\omega(n)} \leqslant d(n) = O(n^{\epsilon'}) \text{ for every } \epsilon' > 0,$$

where $d(n)$ is the number of divisors of n. Choose a positive ϵ' such that $\delta + \epsilon' < (k - 1) + ku_i(1 - \epsilon)$ if (2.17) holds and $k - \delta - \epsilon' > 1$ if (2.18) holds.

From now on, the proof runs similar to that of Theorem 2.5 and we omit the details.

Corollary 3.4.1 — If either (2.16) holds or $k > 1$, the average order of $\Phi_{F^*}^{(k)}(n)$ is $\beta^* n^t$, where

$$\beta^* = \sum_{n=1}^{\infty} \frac{\mu_{F^*}^{(k)}(n)\, \phi_k(n)}{n^{k(t+2)}}.$$

A proof similar to that of Theorem 3.4 gives the following:

Theorem 3.5 — If either (2.16) and (2.22) hold or (2.23) holds, the average order of $\phi_{F^*}^{k,\eta}(n)$ is $\gamma^* n^{kt}$ where

$$\gamma^* = \sum_{n=1}^{\infty} \frac{\mu_{F^*}^{k,\eta}(n)\, \phi(n)}{n^{kt+2}}.$$

Corollary 3.5.1 — Under (2.16), the average order of $\phi_{F*}^{(k)}(n)$ is $\delta^* n^{kt}$, where

$$\delta^* = \sum_{n=1}^{\infty} \frac{\mu_{F*}^{(k)}(n)\,\phi(n)}{n^{kt+2}}.$$

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