Indian J. pure appl. Math., **10**(3) : 307-311, March 1979

ON A CLASS OF COMMUTATORS

S. C. ARORA

Department of Mathematics, Hans Raj College, University of Delhi,
Delhi 110007

AND

RAMESH KUMAR

Department of Mathematics, S. G. T. B. Khalsa College, University of Delhi,
Delhi 110007

(*Received* 20 *May* 1978; *after revision* 17 *July* 1978)

Let $T = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$

be an operator on $H \oplus H$, where H is a separable infinite-dimensional complex Hilbert Space. Following results have been proved :

(i) If $A + D \in X_H$ then $T \in X_{H \oplus H}$.

(ii) If $D = -A$ then $T \in X_{H \oplus H}$.

(iii) If any one of A or D belongs to X_H then $T \in X_{H \oplus H}$.

(iv) The set X_H is strongly dense in $B(H)$, the set of all operators on H.

Let H be a separable infinite-dimensional complex Hilbert space. David (1970) has introduced a class X_H of commutators of bounded linear operators on H :

$$X_H = \{AB - BA : A \text{ is Hermitian}\}.$$

Anderson (1971) has characterized the class X_H as the set of operators with O in the essential numerical range. Our purpose in the present paper is to study some of the properties of the class X_H and generalize some of the results proved by David (1970). Our methods mainly depend on the techniques developed by Arora (1974, 1976).

We start with the following :

Lemma 1 — $A \in X_H$ if and only if $A \oplus A \in X_{H \oplus H}$.

PROOF : Let $A \in X_H$. Then $0 \in W_e(A)$. Hence there exists an orthonormal sequence $\{e_n\}$ in H satisfying

$$(Ae_n, e_n) \to 0 \text{ as } n \to \infty \quad \text{(Anderson 1971).}$$

Let

$$f_n = \left\langle \frac{e_n}{\sqrt{2}}, \frac{e_n}{\sqrt{2}} \right\rangle.$$

Then $\{f_n\}$ is an orthonormal sequence in $H \oplus H$ which satisfies

$$((A \oplus A) f_n, f_n) = \left(\frac{Ae_n}{\sqrt{2}}, \frac{e_n}{\sqrt{2}} \right) + \left(\frac{Ae_n}{\sqrt{2}}, \frac{e_n}{\sqrt{2}} \right) \to 0 \text{ as } n \to \infty.$$

Thus $0 \in W_e(A \oplus A)$ and consequently $A \oplus A \in H_{H \oplus H}$.

To prove the converse we observe that if T_1 and T_2 are operators on H then

$$W(T_1 \oplus T_2) = \text{convex hull of } W(T_1) \text{ and } W(T_2).$$

Now by definition

$$W_e(A \oplus A) = \bigcap_{C} \overline{(W((A \oplus A) + C))},$$

where the intersection is taken over all compact operators C on $H \oplus H$. Let K be any compact operator on H. Then $K \oplus K$ is a compact operator on $H \oplus H$. Hence

$$W_e(A \oplus A) \subset \bigcap_{K} \overline{(W((A \oplus A) + (K \oplus K)))}$$

intersection being taken over all compact operators K on H. But

$$\begin{aligned} W((A \oplus A) + (K \oplus K)) &= W((A + K) \oplus (A + K)) \\ &= \text{convex hull of } W(A + K) \\ &= W(A + K). \end{aligned}$$

Thus

$$W_e(A \oplus A) \subset \bigcap_{K} \overline{W(A + K)} = W_e(A).$$

The lemma follows.

Theorem 1 — If A and B are operators on H such that $A + B \in X_H$, then any operator of the form

$$T = \begin{bmatrix} A & C \\ D & B \end{bmatrix}$$

for arbitrary operators C and D on H belongs to $X_{H \oplus H}$.

PROOF : In view of Lemma I, it suffices to show that

$$T \oplus T \in X_{(H \oplus H) \oplus (H \oplus H)}.$$

By hypothesis, there exists an infinite orthonormal sequence $\{e_n\}$ in H such that

$$((A + B)\, e_n, e_n) \to 0 \text{ as } n \to \infty.$$

Let

$$U_n = \left\langle \frac{e_n}{\sqrt{2}}, 0 \right\rangle, \; V_n = \left\langle 0, \frac{e_n}{\sqrt{2}} \right\rangle.$$

If $f_n = \langle U_n, V_n \rangle$ for each n, one can easily prove that $\{f_n\}$ is an infinite orthonormal sequence of vectors in $(H \oplus H) \oplus (H \oplus H)$. Now

$$\begin{aligned}
([T \oplus T] f_n, f_n) &= ([T \oplus T] \langle U_n, V_n \rangle, \langle U_n, V_n \rangle) \\
&= (TU_n, U_n) + (TV_n, V_n) \\
&= \left(\begin{bmatrix} A & C \\ D & B \end{bmatrix} \left\langle \frac{e_n}{\sqrt{2}}, 0 \right\rangle, \left\langle \frac{e_n}{\sqrt{2}}, 0 \right\rangle \right) \\
&\quad + \left(\begin{bmatrix} A & C \\ D & B \end{bmatrix} \left\langle 0, \frac{e_n}{\sqrt{2}} \right\rangle, \left\langle 0, \frac{e_n}{\sqrt{2}} \right\rangle \right) \\
&= \left(\left\langle \frac{Ae_n}{\sqrt{2}}, \frac{De_n}{\sqrt{2}} \right\rangle, \left\langle \frac{e_n}{\sqrt{2}}, 0 \right\rangle \right) \\
&\quad + \left(\left\langle \frac{Ce_n}{\sqrt{2}}, \frac{Be_n}{\sqrt{2}} \right\rangle, \left\langle 0, \frac{e_n}{\sqrt{2}} \right\rangle \right) \\
&= \left(\frac{Ae_n}{\sqrt{2}}, \frac{e_n}{\sqrt{2}} \right) + \left(\frac{Be_n}{\sqrt{2}}, \frac{e_n}{\sqrt{2}} \right) \\
&= \tfrac{1}{2}([A + B]\, e_n, e_n) \to 0 \text{ as } n \to \infty.
\end{aligned}$$

Hence $T \oplus T \in X_{(H \oplus H) \oplus (H \oplus H)}$ and the theorem follows.

Corollary 1 — Every operator on $H \oplus H$ of the form

$$\begin{bmatrix} A & B \\ C & -A \end{bmatrix}$$

belongs to $X_{(H \oplus H)}$. This generalizes Cor. 5.3 of Brown and Pearcy (1965).

Corollary 2 — If $A + B \in X_H$, then $A \oplus B \in X_{H \oplus H}$. However, on taking $A = 2I$ and $B = -I$, it can be seen that the converse of Cor. 2 is not true.

Theorem 8 of David (1970) proves that if A_1 and A_2 are in X_H, then any operator of the form

$$T = \begin{bmatrix} A_1 & B_1 \\ B_2 & A_2 \end{bmatrix}$$

for arbitrary operators B_1 and B_2 on H, belongs to $X_{H\oplus H}$. This result has also been established by Anderson (1971) by a different approach. We give yet another proof of the same theorem. Since $A_1, A_2 \in X_H$, there exist infinite orthonormal sequences $\{x_n\}$, $\{y_n\}$ in H satisfying

$$(A_1x_n, x_n) \to 0 \text{ and } (A_2y_n, y_n) \to 0 \text{ as } n \to \infty.$$

Let

$$U_n = \left\langle \frac{x_n}{\sqrt{2}}, 0 \right\rangle \text{ and } V_n = \left\langle 0, \frac{y_n}{\sqrt{2}} \right\rangle.$$

It can be easily proved that if $e_n = \langle U_n, V_n \rangle$ for each n, then $\{e_n\}$ is an infinite orthonormal sequence of vectors in $(H \oplus H) \oplus (H \oplus H)$ satisfying

$$([T \oplus T)\, e_n, e_n) \to 0 \text{ as } n \to \infty.$$

Thus $T \oplus T \in X_{(H\oplus H)\oplus(H\oplus H)}$ and by Lemma I, $T \in X_{H\oplus H}$.

We apply our techniques to obtain the following generalized version of Theorem 8 of David (1970).

Theorem 2 — Let

$$T = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

be an operator on $H \oplus H$ where A, B, C and D are operators on H. If any of its diagonal entries belongs to X_H, then $T \in X_{H\oplus H}$.

PROOF : Let $D \in X_H$. Then there exists an infinite orthonormal sequence $\{e_n\}$ of vectors in H satisfying

$$(De_n, e_n) \to 0 \text{ as } n \to \infty.$$

Let

$$f_n = \langle 0, e_n \rangle.$$

Then $\{f_n\}$ is an infinite orthonormal sequence of vectors in $H \oplus H$. We have

$$\begin{aligned}(Tf_n, f_n) &= \left(\begin{bmatrix} A & B \\ C & D \end{bmatrix} \langle 0, e_n \rangle, \langle 0, e_n \rangle \right) \\ &= (\langle Be_n, De_n \rangle, \langle 0, e_n \rangle) \\ &= (De_n, e_n) \to 0 \text{ as } n \to \infty.\end{aligned}$$

Hence $T \in X_{H\oplus H}$. It may, however, be remarked that if $A \in X_H$, then we can choose $f_n = \langle e_n, 0 \rangle$.

Corollary 3 — If N is a normal operator on H having 0 as a limit point of its spectrum and G is a compact operator, then any operator of the form

$$T = \begin{bmatrix} A & B \\ C & N + G \end{bmatrix}$$

for arbitrary operators A, B and C on H, belongs to $X_{H \oplus H}$.

This is a generalization of Theorems 4 and 6 of David (1970) where additional assumption of B and C to be compact has been taken.

Unlike the class of commutators the class X_H is uniformly closed (Anderson 1971). However, we have the following :

Theorem 3 — X_H is strongly dense in $B(H)$.

PROOF : Let $\{f_1, f_2, \ldots, f_n\}$ be a finite set of vectors in H. Let $M = \overline{[f_1, f_2, \ldots, f_n]}$ be the closed linear subspace spanned by $f_1, f_2, \ldots, f_n$. Write

$$H = M \oplus M^{\perp}.$$

Let T be any operator on H. We can write

$$T = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

w.r.t. the decomposition $H = M \oplus M^{\perp}$.

Let

$$S = \begin{bmatrix} A & B \\ C & O \end{bmatrix}.$$

Then $S \in X_H$ as $0 \in X_H$. We also have $Sx = Tx$ for each x in M. This shows that every strong basic neighbourhood of T contains an operator in X_H. Hence the theorem.

ACKNOWLEDGEMENT

The authors are grateful to Dr. B. S. Yadav for his kind guidance in pursuing the work. Thanks are also due to the referee for improving upon the original version of the paper.

REFERENCES

Anderson, J. H. (1971). Commutators and the essential numerical range. Ph.D. Thesis, Indiana University, U.S.A.

Arora, Subhash Chander (1974). On self-commutators. *Math. Nachr.*, **60**, 303-13.

——————(1976). Commutators of operators on Hilbert space. Ph.D. Thesis, Delhi University, Delhi (India).

Brown, A., and Pearcy, C. (1965). Structure of Commutators of operators. *Ann. Math.*, **82**, 112-27.

David, M. (1970). On a certain type of commutators. *J. Math. Mech.*, **19**, 665-80.