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AXISYMMETRIC JET

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On mathematical and physical grounds, it is observed in this paper that Meksyn's method can be well applied to study jet flow problems. As an application, it is illustrated by solving the case of an axisymmetric jet. The results are found to be in good agreement with the solution obtained by Schlichting.

1. INTRODUCTION

Most of the physical problems facing engineers, scientists and research workers today exhibit certain difficulties which prevent exact solutions. Some of the pecularities are : nonlinearity of a differential equation, variable coefficients, boundary conditions at infinity, etc. Even if an exact solution is available, it may not be of much help for physical interpretation of the results. An approximate solution therefore, is always desirable for quick results. Its accuracy, however, may be little less than a numerical solution. Approximate methods, in general, introduce additional assumptions compared to the ones prescribed by a problem. For instance, in integral methods, one assumes *a priori*, a form of velocity and temperature profiles satisfying the boundary conditions. Several approximate methods are available in the literature. A few are : (i) von Karman (1921) and Pohlhausen (1921) integral method, (ii) expansion in power series, i.e. Gortler (1957) new power series method, (iii) step by step method (Gortler 1939), (iv) method of matched and composite expansion (Van Dyke 1964), (v) asymptotic series method due to Meksyn (1961).

In the study of jets one has to face some inherent difficulties; (i) satisfying homogeneous boundary condition at infinity, (ii) physical condition of constancy of flux of momentum or of heat or both. This makes it impossible, for example, to apply the momentum integral method or the technique, of 'reduction to initial value problem' to jets. Traditionally, in case of jets, therefore, either an exact solution (e.g. Schlichting 1939, Bickley 1937) or a numerical solution (e.g. Fujii 1963, Brand and Lahey 1967, Fujii *et al.* 1973) is attempted. Of course there are some works available in which matched and composite expansion technique is used (e.g. Wygnanski and Rotem 1965). In the present paper we demonstrate use of Meksyn's method in jet problems.

In section 2, Meksyn's method is described briefly and it is argued on mathematical and physical grounds that the method can be applied to jet flows. In section 3 this method is applied to axisymmetric jets where exact solutions are already available. In the end, results are compared. It is observed that there is an excellent agreement with the exact solution obtained by Schlichting (1939).

2. The Method : Applicability to Jets

After the successive papers on integration of non-linear ordinary differential equations, Meksyn (1965) developed his widely applied method to find an approximate solution of the boundary layer equations. The power of this method lies in full exploitation of Prandtl's boundary layer assumptions and properties of a flow in equation. Briefly, the method can be described as follows :

For similar solutions of boundary layer equations, for a flat plate, one gets Blasius equation

$$f''' + ff'' = 0 \qquad \ldots(2.1)$$

with boundary conditions

$$\left.\begin{array}{ll} f = f' = 0 & \text{at } \eta = 0 \\ f' \rightarrow 1 & \text{as } \eta \rightarrow \infty \end{array}\right\} \qquad \ldots(2.2)$$

Prandtl's assumption that viscous forces are dominant in a thin layer near a body is equivalent to the statement that the shear stress ($\sim f''$) is rapidly decreasing function of normal coordinate ($\sim \eta$). Equation (2.1), therefore, can be regarded as a linear differential equation in f'' to get its asymptotic value. It makes the solution of a problem comparatively simpler than any other method. $f''(\eta)$ when evaluated from eqn. (2.1) is given by

$$f''(\eta) = e^{-F(\eta)} f''(0) \qquad \ldots(2.3)$$

where

$$F(\eta) = \int_0^{\eta} f(\eta)\, d\eta. \qquad \ldots(2.4)$$

The asymptotic boundary condition of (2.2) yields

$$\int_0^{\infty} e^{-F(\eta)} f''(0)\, d\eta = 1 \qquad \ldots(2.5)$$

which can be evaluated by the method of steepest descent.

It has been proved by Weyl (1941, 1942) that a unique, analytic solution of (2.1) and (2.2) exists in a neighbourhood of the origin $\eta = 0$ with a low radius of convergence. A power series in η will, therefore, be assumed for $f(\eta)$. Boundary conditions (2.2) at $\eta = 0$ require that this power series starts with η^2 term. If it identically satisfies eqn. (2.1), then all the coefficients except one [say the coefficient of η^2, which can be identified with $f''(0)$] remains undetermined. $F(\eta)$ can also be expressed as a power series in η. Series for $F(\eta)$ will start with η^3 and $\eta = 0$ will be a stationary point. Instead of using asymptotic boundary condition (2.2), we evaluate the integral in (2.5) asymptotically and use it to find the unknown coefficient in $f(\eta)$. The main contribution to the integral in (2.5) will come from the region close to $\eta = 0$ (i.e. region near to the wall). Hence the low radius of convergence will be of no importance. Only a few terms of $F(\eta)$ are required. Sometime if needed, Euler's (1913) transformation technique is applied to improve the convergence of the series.

Let us come to jets now. It is well known that Prandtl's boundary layer assumptions are well applicable to describe the motion of jets. These assumptions reduce the equations of motion of a jet to the ordinary boundary layer equations which are simplified with the use of similarity transformations resulting into ordinary non-linear differential equations, for the jet flows. Equation (3.1) is one example. The physical and mathematical roles played by f'' in Blasius equation are replaced by radial velocity g' in axisymmetric jets. $g'(0) \neq 0$ and it decreases to zero as $\eta \to \infty$. Further g' is a function of the exponential type and $G(\eta) = \int_0^\eta \frac{g(\eta)}{\eta} d\eta$ has a stationary point at $\eta = 0$ [These observations are evident by looking at eqns. (3.1) and (3.2)]. Conditions at the wall are replaced by the conditions on the jet axis. It will be observed that the asymptotic condition is satisfied identically and hence will be unable to give us $g'(0)$ (which will be identified with unknown coefficient in $g(\eta)$). But, then, we can invoke the physical condition of constancy of flux to determine it. Thus, over all, it is believed that Meksyn's method is applicable to solve jet problems.

3. Two-Dimensional Jets

We have already discussed about the applicability of the Meksyn's method to jet flow problems. It is illustrated by applying it to the axisymmetric jets where exact solutions (Schlichting 1939) already exist.

The basic equations for a steady axisymmetric jet of an incompressible fluid giving similar solutions along with boundary conditions are well known in standard notation (see Pai 1954) :

$$-\frac{d}{d\eta}\left(\frac{gg'}{\eta}\right) = \frac{d}{d\eta}\left(g'' - \frac{g'}{\eta}\right) \qquad \ldots(3.1)$$

with boundary conditions

$$\left.\begin{aligned} &\text{at } \eta = 0, \quad g = g' = 0 \\ &\eta \to \infty, \quad \frac{g'}{\eta} \to 0. \end{aligned}\right\} \qquad \ldots(3.2)$$

Here g' is related to radial velocity of the jet and dash denotes the differentiation with respect to similarity varaiable η.

The condition of constancy of flux of momentum gives

$$\frac{M_0}{2\pi\mu} = \int_0^\infty \frac{g'^2}{\eta}\, d\eta = \text{constant.} \qquad ...(3.3)$$

Integration of (3.1) along with (3.2) gives

$$g'' - \frac{g'}{\eta} = -\frac{gg'}{\eta}. \qquad ...(3.4)$$

We assume that g is analytical and

$$g(\eta) = \sum_0^\infty a_n \frac{\eta^n}{n!}. \qquad ...(3.5)$$

Let $g''(0) = a$. One finds, with the help of (3.2), (3.4), (3.5), that

$$g(\eta) = \frac{a\eta^2}{2} - \frac{1}{16} a^2\eta^4 + \frac{a^3}{128}\eta^6 - \frac{a^4}{1024}\eta^8. \qquad ...(3.6)$$

Again, integrating (3.4), we have

$$\frac{g'}{\eta} = Ae^{-G(\eta)} \qquad ...(3.7)$$

where

$$G(\eta) = \int_0^\eta \frac{g}{\eta}\, d\eta. \qquad ...(3.8)$$

At $\eta = 0$, g'/η is finite (since $u \propto g'/\eta$) and will be equal to a

$$\therefore \quad a = A. \qquad ...(3.9)$$

Also conditions $g'/\eta = 0$ as $\eta \to \infty$ is satisfied identically by (3.7).

Let $$G(\eta) = \eta^2 \sum_{n=0}^{\infty} c_n\eta^n \qquad ...(3.10)$$

Using (3.8) and (3.10), we find that

$$G(\eta) = \eta^2 \left[\frac{a}{4} - \frac{a^2}{64}\eta^2 + \frac{a^3}{768}\eta^4 \cdots\right]. \qquad ...(3.11)$$

If $\zeta = G(\eta)$, then

$$\eta = \sum_{m=0}^{\infty} \frac{A_m}{m+1} \zeta^{(m+1)/2} \qquad \ldots(3.12)$$

where A_m's will be the coefficient of η^m in the expansion of (see Meksyn 1961, p. 53)

$$\left[\frac{a}{4} - \frac{a^2}{64}\eta^2 + \frac{a^3}{768}\eta^4 \ldots\right]^{-(m+1)/2}$$

Explicitly one gets

$$\eta = \frac{2}{\sqrt{a}} \zeta^{1/2} \left[1 + \frac{1}{8}\zeta + \frac{5}{384}\zeta^2 \ldots\right] \qquad \ldots(3.13)$$

Now coming to the condition of constancy i.e. (3.3) and using (3.7) and (3.9) in this

$$\frac{M_0}{2\pi\mu} = a^2 \int_0^\infty e^{-2\zeta} \eta \, d\eta$$

$$= 2a \int_0^\infty e^{-2\zeta} \left[1 + \frac{\zeta}{2} + \frac{\zeta^2}{8} + \ldots\right] d\zeta$$

$$= a\left[1 + \frac{1}{4}\sqrt{2} + \frac{1}{32}\sqrt{3} + \ldots\right]$$

Considering the summation up to three terms, we obtain

$$a = \frac{M_0}{\pi\mu}\left(\frac{1}{2.6250}\right). \qquad \ldots(3.14)$$

4. Comparison

Available solution in the case of axisymmetric jet is

$$g(\xi) = \frac{\xi^2}{1 + \xi^2/4}, \text{ where } \xi = c\eta$$

$$= 2c^2\left(\frac{\eta^2}{2}\right) - (2c^2)^2 \frac{\eta^4}{16} + (2c^2)^3 \frac{\eta^6}{128} \cdots \qquad \ldots(4.1)$$

If we put $2c^2 = a^2$, then $g(\xi)$ is nothing but the same as (3.6). Now

$$c = (a/2)^{1/2}$$

$$\therefore \quad c^2 = \frac{12}{63} \frac{M_0}{\pi\mu}. \qquad \ldots(4.2)$$

Corresponding value of c^2 given by Pai (1954) is

$$c^2 = \frac{12}{64} \frac{M_0}{\pi\mu}.$$

5. Conclusion and Remarks

Jet flow problems have fulfilled the requirements of Meksyn's method. Comparison discussed above have confirmed the applicability of this method to jet flow problems. Though the method is an approximate one and summations being considered up to three terms only, it has yielded quite satisfactory results.

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