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MAGNETOVISCOUS FLUID COSMOLOGICAL MODELS IN GENERAL RELATIVITY

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Cosmological models representing magnetoviscous fluid with free gravitational field of type D have been obtained. Various physical and geometrical properties have also been obtained.

1. INTRODUCTION

Recently, Roy and Prakash (1976) have obtained some viscous fluid cosmological models of plane symmetry in which a free gravitational field is of Petrov type D. A gravitationally non-degenerate viscous fluid cosmological model has also been derived by Roy and Prakash (1977). In this paper, we have obtained some viscous fluid models in which the material distribution is that of viscous fluid together with an incident magnetic field.

We consider the metric in the form

$$ds^2 = A^2(dx^2 - dt^2) + B^2dy^2 + C^2dz^2 \qquad \ldots(1)$$

where A, B, C are functions of t-alone. The energy momentum tensor for the distribution is considered in the form (Landau and Lifschitz 1963, Lichnerowicz 1967)

$$T_i^j = M_i^j + E_i^j \qquad \ldots(2)$$

where

$$M_i^j = (\epsilon + p)\, v_i v^j + p g_i^j - \eta(v_{i,}^j + v_{,i}^j + v^l v_{i,l} + v_i v^l v_{,l}^j)$$

$$- (\zeta - \tfrac{2}{3}\eta)\, v_{,l}^l\, (g_i^j + v_i v^j) \qquad \ldots(3)$$

$$E_i^j = \bar{\mu}\{ |h|^2 (v_i v^j + \tfrac{1}{2} g_i^j) - h_i h^j\} \qquad \ldots(4)$$

and

$$h_i = \frac{1}{\bar{\mu}}\, {}^*F_{ji} v^j. \qquad \ldots(5)$$

Here p is the isotropic pressure, ϵ the density, η and ζ the two coefficients of viscosity which are taken as constants, h_i the magnetic flux vector, $\bar{\mu}$ the permeability constant, F_{ij} the electromagnetic field tensor and v^i the unit velocity vector satisfying

$$g_{ij}v^i v^j = -1. \qquad ...(6)$$

We assume the coordinates to be co-moving so that $v^1 = v^2 = v^3 = 0$ and $v^4 = \frac{1}{A}$. In the above a comma indicates co-variant differentiation. We take the incident magnetic field to be in the direction of x-axis which leads to F_{23} = constant $= H$ (say)

The field equations

$$R_i^j - \tfrac{1}{2}Rg_i^j + \Lambda g_i^j = -8\pi T_i^j \qquad ...(7)$$

for the line element (1) are

$$\frac{1}{A^2}\left[-\frac{B_{44}}{B} - \frac{C_{44}}{C} - \frac{B_4C_4}{BC} + \frac{A_4B_4}{AB} + \frac{A_4C_4}{AC}\right] - \Lambda$$

$$= 8\pi\left[p - \frac{2\eta A_4}{A^2} - (\zeta - \tfrac{2}{3}\eta)\, v^l_{,l} - \frac{H^2}{2\bar{\mu}B^2C^2}\right], \qquad ...(8)$$

$$\frac{1}{A^2}\left[-\frac{C_{44}}{C} - \frac{A_{44}}{A} + \frac{A_4^2}{A^2}\right] - \Lambda$$

$$= 8\pi\left[p - \frac{2\eta B_4}{AB} - (\zeta - \tfrac{2}{3}\eta)\, v^l_{,l} + \frac{H^2}{2\bar{\mu}B^2C^2}\right], \qquad ...(9)$$

$$\frac{1}{A^2}\left[-\frac{B_{44}}{B} - \frac{A_{44}}{A} + \frac{A_4^2}{A^2}\right] - \Lambda$$

$$= 8\pi\left[p - \frac{2\eta C_4}{AC} - (\zeta - \tfrac{2}{3}\eta)\, v^l_{,l} + \frac{H^2}{2\bar{\mu}B^2C^2}\right], \qquad ...(10)$$

$$\frac{1}{A^2}\left[\frac{A_4B_4}{AB} + \frac{A_4C_4}{AC} + \frac{B_4C_4}{BC}\right] + \Lambda = 8\pi\left[\epsilon + \frac{H^2}{2\bar{\mu}B^2C^2}\right]. \qquad ...(11)$$

The suffix 4 after A, B, C denotes ordinary differentiation with respect to t. Equations (8) – (11) are four equations in five unknowns A, B, C, ϵ and p. For the complete determination of this set, we need an extra condition. We assume that the gravitational field is of Petrov type D, the degeneracy being in the direction of y and z axes. This requires that

$$C^{12}_{12} = C^{13}_{13}. \qquad ...(12)$$

The condition (12) is identically satisfied if $B = C$. Therefore we choose A, B, C to be unequal. From condition (12), we have

$$\frac{B_{44}}{B} - \frac{C_{44}}{C} + \frac{2A_4}{A}\left(\frac{C_4}{C} - \frac{B_4}{B}\right) = 0. \quad \text{...(13)}$$

From eqns. (8) and (9), we have

$$\left(\frac{A_4}{A}\right)_4 + \frac{A_4}{A}\left(\frac{B_4}{B} + \frac{C_4}{C}\right) - \frac{B_{44}}{B} - \frac{B_4 C_4}{BC}$$
$$= 16\pi\eta A\left(\frac{B_4}{B} - \frac{A_4}{A}\right) - \frac{8\pi H^2 A^2}{\bar{\mu} B^2 C^2}. \quad \text{...(14)}$$

From eqns. (9) and (10), we have

$$\frac{B_{44}}{B} - \frac{C_{44}}{C} = 16\pi\eta A\left(\frac{C_4}{C} - \frac{B_4}{B}\right). \quad \text{...(15)}$$

Equations (13) and (15) lead to

$$\frac{2A_4}{A}\left(\frac{B_4}{B} - \frac{C_4}{C}\right) = 16\pi\eta A\left(\frac{C_4}{C} - \frac{B_4}{B}\right). \quad \text{...(16)}$$

Since $B \neq C$, we have

$$\frac{A_4}{A} = -\,8\pi\eta A. \quad \text{...(17)}$$

Equation (17) on integration leads to

$$A = (8\pi\eta t + a)^{-1} \quad \text{...(18)}$$

a being a constant of integration.

From eqns. (15) and (18), we have

$$C^2\left(\frac{B}{C}\right)_4 = \frac{b^2}{(8\pi\eta t + a)^2} \quad \text{...(19)}$$

b being a constant of integration. From eqns. (14) and (18), we have

$$\left(\frac{8\pi\eta}{8\pi\eta t + a}\right)^2 + \frac{8\pi\eta}{8\pi\eta t + a}\left(\frac{B_4}{B} + \frac{C_4}{C}\right) + \frac{16\pi\eta}{8\pi\eta t + a}\cdot\frac{B_4}{B}$$
$$+ \frac{B_{44}}{B} + \frac{B_4 C_4}{BC} - \frac{8\pi H^2}{\bar{\mu} B^2 C^2 (8\pi\eta t + a)^2} = 0. \quad \text{...(20)}$$

Putting $BC = \mu$ and $\frac{B}{C} = \nu$, in eqns. (19) and (20), we have

$$\frac{\nu_4}{\nu} = \frac{b^2}{(8\pi\eta t + a)^2\,\mu} \quad \text{...(21)}$$

and

$$\left(\frac{8\pi\eta}{8\pi\eta t + a}\right)^2 + \frac{8\pi\eta}{8\pi\eta t + a} \cdot \frac{\mu_4}{\mu} + \frac{8\pi\eta}{8\pi\eta t + a}\left(\frac{\mu_4}{\mu} + \frac{\nu_4}{\nu}\right)$$
$$+ \frac{\mu_{44}}{2\mu} + \frac{\nu_{44}}{2\nu} + \frac{\mu_4\nu_4}{2\mu\nu} - \frac{\nu_4^2}{2\nu^2} - \frac{8\pi H^2}{\bar{\mu}(8\pi\eta t + a)^2\,\mu^2} = 0. \qquad ...(22)$$

From eqns. (21) and (22), we have

$$\left(\frac{8\pi\eta t + a}{8\pi\eta}\right)^2 \mu\mu_{44} + 4\mu\mu_4\left(\frac{8\pi\eta t + a}{8\pi\eta}\right) + 2\mu^2 = N^2 \qquad ...(23)$$

where

$$N^2 = \frac{H^2}{4\pi\bar{\mu}\eta^2}. \qquad ...(24)$$

Putting $\frac{8\pi\eta t + a}{8\pi\eta} \cdot \frac{d\mu}{dt} = \frac{d\mu}{d\tau}$ in eqn. (23), we have

$$\frac{d^2\mu}{d\tau^2} + 3\frac{d\mu}{d\tau} + 2\mu = \frac{N^2}{\mu}. \qquad ...(25)$$

The general solution of eqn. (25) is not possible. However, if we put $\frac{d\mu}{d\tau} = \frac{1}{f(\mu)}$, eqn. (25) leads to

$$f' = 3f^2 + \left(2\mu - \frac{N^2}{\mu}\right)f^3 \qquad ...(26)$$

which is Abel's equation of the first kind. Here a prime denotes differentiation with respect to μ. From eqns. (26) and (21), we determine the value of μ and ν respectively. In the following, we obtain some particular solutions.

Particular Cases

Case (i) : μ = constant

From eqn. (25), we have

$$\mu = \frac{H}{\eta\sqrt{8\bar{\mu}\pi}}. \qquad ...(27)$$

Equations (21) and (27) lead to

$$\nu = \delta \exp\left[\frac{-b^2\sqrt{8\bar{\mu}\pi}}{8\pi H(8\pi\eta t + a)}\right] \qquad ...(28)$$

δ being a constant of integration. Therefore

$$B^2 = \mu\nu = \frac{H\delta}{\eta\sqrt{8\bar{\mu}\pi}} \exp\left[\frac{-b^2\sqrt{8\bar{\mu}\pi}}{8\pi H(8\pi\eta t + a)}\right] \qquad ...(29)$$

and

$$C^2 = \frac{\mu}{\nu} = \frac{H}{\delta\eta\sqrt{8\bar{\mu}\pi}} \exp\left[\frac{b^2\sqrt{8\bar{\mu}\pi}}{8\pi H(8\pi\eta t + a)}\right]. \quad ...(30)$$

After suitable transformations of co-ordinates the metric reduces to the form

$$ds^2 = T^{-2}dx^2 - \frac{T^{-2}}{64\pi^2\eta^2}dT^2 + \frac{H\delta}{\eta\sqrt{8\bar{\mu}\pi}} \exp\left[\frac{-b^2\sqrt{8\bar{\mu}\pi}}{8\pi HT}\right] dy^2$$

$$+ \frac{H}{\delta\eta\sqrt{8\bar{\mu}\pi}} \exp\left[\frac{b^2\sqrt{8\bar{\mu}\pi}}{8\pi HT}\right] dz^2. \quad ...(31)$$

Case (ii)

$$A = KB^2 \quad ...(32)$$

where K is a constant. From eqns. (17) and (32), we have

$$-\frac{2B_4}{B^3} = 8\pi\eta K. \quad ...(33)$$

Equation (33) on integration leads to

$$B^2 = \frac{1}{(8\pi\eta Kt + \beta)} \quad ...(34)$$

β being a constant of integration. From eqns. (32) and (34), we have

$$A^2 = \frac{K^2}{(8\pi\eta Kt + \beta)^2}. \quad ...(35)$$

From eqns. (14), (34) and (35), we have

$$2CC_4 + \frac{8\pi\eta K}{8\pi\eta Kt + \beta} C^2 = \frac{8\pi H^2 K^2}{\bar{\mu}} \quad ...(36)$$

Equation (36) on integration leads to

$$C^2 = \frac{H^2K(8\pi\eta Kt + \beta)}{2\bar{\mu}\eta} + \frac{\gamma}{(8\pi\eta Kt + \beta)} \quad ...(37)$$

γ being a constant of integration. After suitable transformations of co-ordinates, the metric reduces to the form

$$dS^2 = T^{-2}dX^2 - \frac{T^{-2}}{64\pi^2\eta^2}dT^2 + T^{-1}dy^2$$

$$+\left[\frac{KH^2T}{2\bar{\mu}\eta} + \gamma T^{-1}\right] dz^2. \quad ...(38)$$

Some Physical and Geometrical Features

The pressure and density for the model (31) are given by

$$8\pi p = \frac{\pi\eta^2 b^2 \,[12HT\sqrt{8\bar{\mu}\pi} - \bar{\mu}b^2]}{2H^2T^2} - 96\pi^2\eta^2 - 64\pi^2\eta(\zeta - \tfrac{2}{3}\eta) - \Lambda \quad ...(39)$$

and

$$8\pi\epsilon = -\frac{2\pi\bar{\mu}b^4\eta^2}{H^2T^2} - 32\pi^2\eta^2 + \Lambda. \quad ...(40)$$

The scalar of expansion θ calculated for the flow vector is given by

$$\theta = -8\pi\eta. \quad ...(41)$$

The rotation ω is identically zero and the non-vanishing components of the shear tensor σ_{ij} are given by

$$\sigma_{11} = -\frac{16\pi\eta}{3T^2}, \quad ...(42)$$

$$\sigma_{22} = \frac{H\delta}{\eta\sqrt{8\bar{\mu}\pi}}\left(\frac{b^2\eta\sqrt{8\bar{\mu}\pi}}{2HT} + \frac{8\pi\eta}{3}\right)\exp\left[\frac{-b^2\sqrt{8\bar{\mu}\pi}}{8\pi HT}\right], \quad ...(43)$$

$$\sigma_{33} = \frac{H}{\delta\eta\sqrt{8\bar{\mu}\pi}}\left(\frac{8\pi\eta}{3} - \frac{b^2\eta\sqrt{8\bar{\mu}\pi}}{2HT}\right)\exp\left[\frac{b^2\sqrt{8\bar{\mu}\pi}}{8\pi HT}\right]. \quad ...(44)$$

The model has to satisfy the reality conditions (Ellis 1971)

(i) $(\epsilon + p) > 0$ and (ii) $(\epsilon + 3p) > 0$.

The condition (i) leads to

$$\frac{3\eta b^2\sqrt{8\bar{\mu}\pi}}{64\pi H(\zeta + \frac{4}{3}\eta)} - \frac{b^2\sqrt{2\pi\eta\bar{\mu}\,\{9\eta - 20(\zeta + \frac{4}{3}\eta)\}}}{32\pi H(\zeta + \frac{4}{3}\eta)} < T < \frac{3\eta b^2\sqrt{8\bar{\mu}\pi}}{64\pi H(\zeta + \frac{4}{3}\eta)} + \frac{b^2\sqrt{2\pi\eta\bar{\mu}\,\{9\eta - 20(\zeta + \frac{4}{3}\eta)\}}}{32\pi H(\zeta + \frac{4}{3}\eta)} \quad ...(45a)$$

and

$$\zeta > \frac{53}{60}\eta. \quad ...(45b)$$

The condition (ii) leads to

$$36\pi\eta^2 b^2 HT - 7\pi\bar{\mu}\eta^2 b^4 > 2H^2T^2\,\{192\pi^2\eta(\zeta + \eta) + 2\Lambda\} \quad ...(46)$$

which gives condition on Λ. The flow vector is geodetic and the model represents contracting, shearing but non-rotating universe.

The pressure and density for the model (38) are given by

$$8\pi p = \frac{2\eta}{(H^2KT^2 + 2\bar{\mu}\eta\gamma)}[(2\pi H^2KT^2 - 4\pi\eta\bar{\mu}\gamma)\{8\pi(\zeta - \tfrac{2}{3}\eta) + 16\pi\eta\} - 4\pi H^2T^2] - 112\pi^2\eta^2 - 96\pi^2\eta(\zeta - \tfrac{2}{3}\eta) - \Lambda \quad \ldots(47)$$

and

$$8\pi\epsilon = 16\pi^2\eta^2 - \frac{2\eta}{(H^2KT^2 + 2\bar{\mu}\eta\gamma)} \times [12\pi\eta(2\pi KH^2T^2 - 4\pi\eta\bar{\mu}\gamma) + 4\pi H^2T^2] + \Lambda. \quad \ldots(48)$$

The scalar of expansion θ calculated for the flow vector is given by

$$\theta = \frac{2\eta\,[2\pi KH^2T^2 - 4\pi\eta\bar{\mu}\gamma]}{[H^2KT^2 + 2\bar{\mu}\eta\gamma]} - 12\pi\eta. \quad \ldots(49)$$

The rotation ω is identically zero and the non-vanishing components of the shear tensor σ_{ij} are given by

$$\sigma_{11} = -\frac{4\pi\eta}{K^2T^2} - \frac{(2\pi KH^2T^2 - 4\pi\eta\bar{\mu}\gamma)}{3K^2T^2(H^2KT^2 + 2\bar{\mu}\eta\gamma)}. \quad \ldots(50)$$

$$\sigma_{22} = -\frac{4\pi\eta}{T} + \frac{4\pi\eta K}{T^2} - \frac{K(2\pi KH^2T^2 - 4\pi\eta\bar{\mu}\gamma)}{3T^2(H^2KT^2 + 2\bar{\mu}\eta\gamma)}, \quad \ldots(51)$$

$$\sigma_{33} = \frac{2T}{3K}\left[\frac{2\pi K^2H^2}{\bar{\mu}} - \frac{4\pi\eta K\gamma}{T}\right] + 4\pi\eta\left[\frac{H^2KT}{2\bar{\mu}\eta} + \frac{\gamma}{T}\right]. \quad \ldots(52)$$

The model has to satisfy the reality conditions (Ellis 1971)

(i) $(\epsilon + p) > 0$

and

(ii) $(\epsilon + 3p) > 0$.

The condition (i) leads to

$$\frac{2\eta\bar{\mu}\gamma}{KH^2} < T^2 < \frac{4\left(\zeta + \frac{5}{24}\eta\right)}{H^2}. \quad \ldots(53)$$

The condition (ii) leads to

$$2\eta\,[(2\pi KH^2T^2 - 4\pi\eta\bar{\mu}\gamma)\{24\pi(\zeta + \tfrac{5}{6}\eta)\} - 16\pi^2H^2T^2] > [288\pi^2\eta(\zeta + \tfrac{4}{9}\eta) + 2\Lambda]\,(H^2KT^2 + 2\eta\bar{\mu}\gamma) \quad \ldots(54)$$

which gives condition on Λ. The flow vector is geodetic and the model represents expanding, shearing but non-rotating universe.

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