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OPERATIONAL DERIVATION OF GENERATING FUNCTIONS OF A GENERALIZED FUNCTION

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Recently, Patil and Thakare (1976) have introduced a generalized function

$P_n^{\alpha}(x; r, s, p, k, \lambda)$ defined by

$$P_n^{\alpha}(x; r, s, p, k, \lambda) = x^{-\alpha} \exp(px^r)\, \theta^n x^{\alpha+sn} \exp(-px^r)$$

where

$$\theta = \lambda x^k + x^{k+1}D,\ D \equiv d/dx.$$

In the present paper we obtain two generating relations for these polynomials by operational technique.

§1. In a recent paper Patil and Thakare (1976) studied the differential operator

$$\theta = x^k(\lambda + xD),\ D \equiv \frac{d}{dx} \qquad \text{...(1.1)}$$

which led them to introduce the polynomials $P_n^{\alpha}(x, r, s, p, k, \lambda)$ by means of the generalized nth differential formula (Patil and Thakare 1975a)

$$P_n^{\alpha}(x, r, s, p, k, \lambda) \equiv P_n^{\alpha}(x) = x^{-\alpha} \exp(px^r)\, \theta^n(x^{\alpha+n} \exp(-px^r)) \qquad \text{...(1.2)}$$

The study of these polynomials was further continued by Patil and Thakare (1975b).

It is evident from the nature of the parameters that almost all the orthogonal and non-orthogonal polynomials can be derived from (1.2).

The object of the present paper is to establish two generating functions for these polynomials by operational technique.

§2. In our analysis we shall make use of a number of known results which we mention here for ready reference :

$$\theta^n = x^{kn} \prod_{j=0}^{n-1} (\delta + \lambda + jk),\ \delta \equiv xD \qquad \text{...(2.1)}$$

$$a^{\delta} f(x) = f(ax) \qquad ...(2.2)$$

$$(1+t)^{-\delta-\alpha} f(x) = \sum_{n=0}^{\infty} \frac{(-t)^n}{n!} (\delta+\alpha)_n f(x) \qquad ...(2.3)$$

and

$$(1+t)^{-\delta-\alpha} = (1+t)^{-\delta} (1+t)^{-\alpha}. \qquad ...(2.4)$$

From (1.2) and (2.1) we arrive at the operational representation of the polynomials $P_n^{\alpha-sn}(x)$ as

$$x^{-sn} P_n^{\alpha-sn}(x) = (x)^{-\alpha} \exp(px^r)\, x^{kn} \prod_{j=0}^{n-1} (\delta+\lambda+jk)\,(x^{\alpha} \exp(-px^r)).$$

Hence,

$$\sum_{n=0}^{\infty} x^{-(k+s)n} P_n^{\alpha-sn}(x) \frac{t^n}{n!}$$

$$= x^{-\alpha} \exp(px^r) \sum_{n=0}^{\infty} \frac{t^n}{n!} \prod_{j=0}^{n-1} (\delta+\lambda+jk)\,(x^{\alpha} \exp(-px^r))$$

$$= x^{-\alpha} \exp(px^r) \sum_{n=0}^{\infty} \frac{(kt)^n}{n!} \left(\frac{\delta+\lambda}{k}\right)_n (x^{\alpha} \exp(-px^r))$$

$$= x^{-\alpha} \exp(px^r)\,(1-kt)^{-\lambda/k} [(1-kt)^{-1/k}]^{\delta} (x^{\alpha} \exp(-px^r))$$

$$= x^{-\alpha} \exp(px^r)\,(1-kt)^{-\lambda/k} [x(1-kt)^{-1/k}]^{\alpha}$$

$$\times \exp[-p\{x(1-kt)^{-1/k}\}^r].$$

We have therefore obtained a generating relation for $P_n^{\alpha-sn}(x)$,

$$\sum_{n=0}^{\infty} x^{-(k+s)n} P_n^{\alpha-sn}(x) \frac{t^n}{n!} = (1-kt)^{-(\alpha+\lambda)/k}$$

$$\times \exp[px^r\{1-(1-kt)^{-r/k}\}]. \qquad ...(2.5)$$

Replacing t by (tx^{k+s}) in (2.5), we obtain

$$\sum_{n=0}^{\infty} P_n^{\alpha-sn}(x) \frac{t^n}{n!} = (1-kt\,x^{k+s})^{-(\alpha+\lambda)/k}$$

$$\times \exp[px^r\{1-(1-kt\,x^{k+s})^{-r/k}\}]. \qquad ...(2.6)$$

From (1.2) and (2.1), we obtain

$$x^{-(k+s)(n+m)}\, P_{n+m}^{\alpha-s(n+m)}(x) = x^{-\alpha} \exp(px^r)$$

$$\times \prod_{j=0}^{n+m-1} (\delta + \lambda + jk)\,(x^\alpha \exp(-px^r))$$

$$= x^{-\alpha} \exp(px^r) \prod_{j=m}^{n+m-1} (\delta + \lambda + jk) \prod_{j=0}^{m-1} (\delta + \lambda + jk)(x^\alpha \exp(-px^r))$$

$$= x^{-\alpha} \exp(px^r) \prod_{j=m}^{n+m-1} (\delta + \lambda + jk)\, x^{\alpha-(k+s)m} \exp(-px^r)\, P_m^{\alpha-sm}(x)$$

$$= x^{-\alpha} \exp(px^r)\, k^n \left(\frac{\delta+\lambda}{k} + m\right)_n x^{\alpha-(k+s)m} \exp(-px^r)\, P_m^{\alpha-sm}(x).$$

Therefore,

$$\sum_{n=0}^{\infty} x^{-(k+s)(n+m)}\, P_{n+m}^{\alpha-s(n+m)}(x) \frac{t^n}{n!}$$

$$= x^{-\alpha} \exp(px^r) \sum_{n=0}^{\infty} \frac{(kt)^n}{n!} \left(\frac{\delta+\lambda}{k} + m\right)_n x^{\alpha-(k+s)m} \exp(-px^r)\, P_m^{\alpha-sm}(x)$$

$$= x^{-\alpha} \exp(px^r)\,(1-kt)^{-(\lambda/k+m)}\,[(1-kt)^{-1/k}]^{\delta}$$

$$\times\, x^{\alpha-(k+s)m} \exp(-px^r)\, P_m^{\alpha-sm}(x)$$

$$= x^{-\alpha} \exp(px^r)\,(1-kt)^{-(\lambda/k+m)}\,[x(1-kt)^{-1/k}]^{\alpha-(k+s)m}$$

$$\times \exp[-p\,\{x(1-kt)^{-1/k}\}^r]\, P_m^{\alpha-sm}(x(1-kt)^{-1/k}).$$

Thus,

$$\sum_{n=0}^{\infty} x^{-(k+s)n}\, P_{n+m}^{\alpha-s(n+m)}(x) \frac{t^n}{n!} = (1-kt)^{-(\lambda+\alpha-sm)/k}$$

$$\times \exp[px^r\{1-(1-kt)^{-r/k}\}]\, P_m^{\alpha-sm}(x(1-kt)^{-1/k}). \qquad \ldots(2.7)$$

Replacing t by $(x^{k+s}t)$ in (2.7) we obtain another generating relation for polynomials $P_{n+m}^{\alpha-s(n+m)}(x)$,

$$\sum_{n=0}^{\infty} P_{n+m}^{\alpha-s(n+m)}(x) \frac{t^n}{n!} = (1-kt\,x^{k+s})^{-(\lambda+\alpha-sm)/k}$$

$$\times \exp[px^r \{1-(1-kt\,x^{k+s})^{-r/k}\}]$$

$$\times P_m^{\alpha-sm}(x(1-kt\,x^{k+s})^{-1/k}). \qquad \ldots(2.8)$$

The results (2.6) and (2.8) are also given by Patil and Thakare (1977) by a different method.

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