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DECOMPOSITION OF BERWALD'S CURVATURE TENSOR FIELD IN GENERALIZED 2-RECURRENT FINSLER SPACE

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The decomposition of curvature tensor field in Riemannian space was first discussed by Takano (1967). This idea was introduced in second order recurrent Finsler space by Pande and Khan (1974) by taking different curvature tensor field. The present paper deals the decomposition of Berwald's curvature tensor field in generalized 2-recurrent Finsler space. The various properties of decomposed vector and tensor fields have been obtained.

1. INTRODUCTION

Let us consider an n-dimensional Finsler space F_n equipped with $2n$ arguments x^i and $\dot{x}^i$ in which the metric is defined by the fundamental function $F(x, \dot{x})$. The function $F(x, \dot{x})$ is positively homogeneous of degree one in its directional arguments. The fundamental metric tensor is given by

$$g_{ij}(x, \dot{x}) \overset{def}{=} \tfrac{1}{2}\, \dot{\partial}_i \dot{\partial}_j F^2(x, \dot{x})^*. \qquad ...(1.1)$$

The metric tensor is positively homogeneous of degree zero in its directional argument and symmetric in the lower indices i and j. The covariant and contravariant components of the metric tensor satisfy the following relation :

$$g_{ij} g^{kj} = \delta^k_i \qquad ...(1.2)$$

where δ^k_i is Kronecker delta.

The Berwald curvture tensor and their contracted forms are (Rund 1959)

$$H^i_{jk} = 2\partial_{[k}\dot{\partial}_{j]}\, G^i + 2G^i_{r[k}\, \dot{\partial}^{\dagger}_{j]}\, G^r \qquad ...(1.3)$$

$$H^i_{jkh} = 2\partial_{[h} G^i_{k]j} + 2G^r_{j[h} G^i_{k]r} + 2G^i_{rj[h}\, \dot{\partial}_{k]} G^r \qquad ...(1.4)$$

$^*\dot{\partial}_i \equiv \frac{\partial}{\partial x^i}$ and $\partial_i \equiv \frac{\partial}{\partial \dot{x}^i}$

$^\dagger 2A_{(hk)} = A_{hk} + A_{kh}$ and $2A_{[hk]} = A_{hk} - A_{kh}$

and

$$\text{(i)} \quad H_{ij} = H^r_{ijr} = \dot{\partial}_i H_j, \text{ (ii) } 2H_{[jk]} = H^r_{rkj},$$

$$\text{(iii)} \quad H_j = H^r_{jr}, \text{ (iv) } H = H^i_i\,(n-1). \qquad \ldots(1.5)$$

The curvature tensor satisfy the following identities (Rund 1959) :

$$\text{(i)} \quad H^i_{jkh} = -H^i_{jhk}, \text{ (ii) } H^i_k\,\dot{x}^k = 0,$$

$$\text{(iii)} \quad H_{jk}\dot{x}^j = H_k, \text{ (iv) } H^i_{jkh}\,\dot{x}^j = H^i_{kh},$$

$$\text{(v)} \quad \dot{\partial}_r H^r_j\,\dot{x}^j + (n-1)\,H = 0. \qquad \ldots(1.6)$$

The commutation formulae for Berwald's curvature tensor fields are

$$T^i_{j(h)(k)} - T^i_{j(k)(h)} = -\dot{\partial}_r T^i_j\,H^r_{hk} + T^r_j\,H^i_{rhk} - T^i_r\,H^r_{jhk} \qquad \ldots(1.7)$$

and

$$(\dot{\partial}_k T^i_j)_{(h)} - \dot{\partial}_k\,(T^i_{j(h)}) = T^i_r\,G^r_{jkh} - T^r_j\,G^i_{rkh}. \qquad \ldots(1.8)$$

Definition 1.1 — A Finsler space F_n is said to be recurrent Finsler space of first order, if the curvature tensor $H^i_{jkh}\,(x, \dot{x})$ satisfies the relation (Sinha 1972)

$$H^i_{jkh(l)} = v_l H^i_{jhk} \qquad \ldots(1.9)$$

where $v_{l(x)}$ is a vector field depending only upon the positional arguments.

Definition 1.2 — In an F_n the Berwald curvature tensor field $H^i_{jkh}\,(x, \dot{x})$ satisfies the relation (Pande and Khan 1978)

$$H^i_{jkh(l)(m)} = U_m H^i_{jkh(l)} + d_{lm} H^i_{jkh} \qquad \ldots(1.10)$$

and

$$H^i_{jkh} \neq 0, \qquad \ldots(1.11)$$

where $U_m(x)$ and $d_{lm}(x, \dot{x})$ are associate non-null recurrence vector and non-zero tensor fields, respectively, then it is said to be generalized 2-recurrent Finsler space and is denoted by 2-RF_n.

2. Decomposition of Berwald's Curvature Tensor Field H^i_{jkh}

We consider the decomposition of the Berwald's curvature tensor field in the following manner (Pande and Khan 1973)

$$H^i_{jkh} = X^i_j \phi_{kh} \qquad \text{...(2.1)}$$

where $\phi_{kh}(x, \dot{x})$ and $X^i_j(x, \dot{x})$ are two tensor fields such that

$$X^i_j v_i = p_j$$

where p_j is decomposed non-null vector field.

Transvecting eqn. (2.1) by $\dot{x}^j$ and using the relation (1.6 (iv)), we get

$$H^i_{kh} = X^i_j \phi_{kh}\dot{x}^j. \qquad \text{...(2.2)}$$

Let us suppose that the vector v_i and the tensor X^i_j are recurrent, which are given by

$$v_{i(m)} = \lambda_m v_i \qquad \text{...(2.3)}$$

and

$$X^i_{j(m)} = \mu_m X^i_j \qquad \text{...(2.4)}$$

where $\lambda_m(x)$ and $\mu_m(x)$ are non-zero recurrent vector fields. In view of which we get

$$p_{j(m)} = (\lambda_m + \mu_m)\, p_j \qquad \text{...(2.5)}$$

and

$$\phi_{kh(m)} = (v_m - \mu_m)\, \phi_{kh}. \qquad \text{...(2.6)}$$

Thus, we have the following theorems :

Theorem 2.1 — $\phi_{kh}(x, \dot{x})$ is a tensor of 2-RF_n iff

$$\mu_l(U_m + \mu_m) = \mu_{l(m)} + v_m\mu_l + v_l\mu_m. \qquad \text{...(2.7)}$$

Proof : Differentiating (2.1) covariantly with respect to x^l and x^m successively, we get

$$H^i_{jkh(l)(m)} = X^i_{j(l)(m)}\phi_{kh} + X^i_{j(l)}\phi_{kh(m)} + X^i_{j(m)}\phi_{kh(l)} + X^i_j \phi_{kh(l)(m)}. \qquad \text{...(2.8)}$$

By virtue of eqns. (1.9), (1.10) and using the relations (2.1), (2.4) and (2.6) in eqn. (2.8), we get

$$U_m\phi_{kh(l)} + d_{lm}\phi_{kh} = \phi_{kh(l)(m)} + \mu_{l(m)} + v_m\mu_l + v_l\mu_m - \mu_l\mu_m - U_m\mu_l. \qquad ...(2.9)$$

Now, if we suppose that the condition (2.7) is true then

$$U_m\phi_{kh(l)} + d_{lm}\phi_{kh} = \phi_{kh(l)(m)}. \qquad ...(2.10)$$

Conversely if (2.10) holds, from eqn. (2.9) we have

$$(\mu_{l(m)} + \mu_l v_m + \mu_m v_l - \mu_l\mu_m - \mu_l U_m)\,\phi_{kh} = 0 \qquad ...(2.11)$$

which proves the sufficient part of the theorem.

Theorem 2.2 — In 2-RF_n, the skew-symmetric part of the associate recurrence tensor field d_{lm} satisfies the relation

$$(d_{lm} - d_{ml})_{(n)} = (\mu_{l(m)} - \mu_{m(l)})_{(n)} + (U_l v_m - U_m v_l)_{(n)} + (\alpha_{(n)} + \alpha v_n)\,\phi_{lm} - \alpha\mu_n\phi_{lm} \qquad ...(2.12a)$$

where the scalar function (Pande and Khan 1974)

$$\alpha \overset{def}{=} \dot{\partial}_r X^r_t\, \dot{x}^t - X^r_r\,. \qquad ...(2.12b)$$

PROOF : Commutating eqn. (2.9) with respect to indices l and m and applying the commutation formula (1.7) and using the relations (1.5 (i), (ii)), (2.1), (2.2), (2.6) and the fact that $\phi_{kh} = -\phi_{hk}$, we get

$$\{(d_{lm} - d_{ml}) + (\mu_{m(l)} - \mu_{l(m)}) + (v_l U_m - U_l v_m)\}\,\phi_{kh} = (-\dot{\partial}_r\phi_{kh} X^r_t\,\dot{x}^t + X^r_r\,\phi_{hk})\,\phi_{lm}. \qquad ...(2.13)$$

Differentiating (2.13) covariantly with respect to x^n and using eqns. (2.6), (2.13) and the fact that $-\dot{\partial}_r\phi_{kh} X^r_t\,\dot{x}^t = \dot{\partial}_r X^r_t\,\phi_{kh}\dot{x}^t$, we obtain the required result.

Theorem 2.3 — Under the decomposition (2.1) the associate recurrence tensor field d_{lm} and the decomposed tensor field ϕ_{lm} satisfy the following relation :

$$[d_{[lm(n)]} + \mu_{[l(m)(n)]} + d_{[lm}v_{n]} + \mu_{[l(m)}v_{n]} + v_{[l}U_{m(n)]} + \mu_{[l}U_{n(m)]} + d_{[ln}U_{m]} + d_{[ln}\mu_{m]} + \mu_{[l(n)}\mu_{m]}] = (\beta_l\phi_{mn} + \beta_m\phi_{nl} + \beta_n\phi_{lm}) \qquad ...(2.14a)$$

where β_l is given by (Pande and Khan 1974)

$$\beta_l = \dot{\partial}_r\mu_l X^r_t\,\dot{x}^t + \mu_l(X^r_r - \dot{\partial}_r X^r_t\,\dot{x}^t). \qquad ...(2.14b)$$

PROOF : Differentiating (2.9) covariantly with respect to x^n and commutating the obtained result in m and n and using eqn. (2.10), we get

$$\phi_{kh(l)(m)(n)} - \phi_{kh(l)(n)(m)} = (d_{lm(n)} - d_{ln(m)})\,\phi_{kh}$$
$$+ (d_{lm}\phi_{kh(n)} - d_{ln}\phi_{kh(m)}) + \phi_{kh(l)}(U_{m(n)} - U_{n(m)})$$
$$+ (U_m d_{ln} - U_n d_{lm})\,\phi_{kh} + (I_{lm(n)} - I_{ln(m)})\,\phi_{kh}$$
$$+ I_{lm}\phi_{kh(n)} - I_{ln}\phi_{kh(m)} \qquad \ldots(2.15)$$

where

$$I_{lm} = \mu_{l(m)} - \mu_l\mu_m - U_l\mu_m + \nu_m\mu_l + \nu_l\mu_m.$$

Applying the commutation formula (1.7) in the above equation and with the help of the relations (1.5 (i), (ii)), (2.1), (2.2), (2.6) and the fact that $\phi_{kh} = -\phi_{hk}$, (2.15) takes the form

$$\{(d_{lm(n)} - d_{ln(m)}) + (d_l\nu_{mn} - d_{ln}\nu_m) + (d_{ln}\mu_m - d_{lm}\mu_n)$$
$$+ \nu_l(U_{m(n)}U_{n(m)}) + \mu_l(U_{n(m)} - U_{m(n)}) + (U_m d_{ln} - U_n d_{lm})$$
$$+ (I_{lm(n)} - I_{ln(m)}) + I_{lm}(\nu_n - \mu_n)\}\,\phi_{kh}$$
$$= \{(\dot{\partial}_r\mu_l)\,\phi_{kh} - (\nu_l - \mu_l)\,\dot{\partial}_r\phi_{kh}\}\,X^r_t\,\dot{x}^t\phi_{mn}$$
$$- \{(\nu_l - \mu_l)\,X^r_r + (\nu_r - \mu_r)\,X^r_l\}\,\phi_{kh}\,\phi_{lm} \qquad \ldots(2.16)$$

where ν_i is taken as independent of direction.

Adding the expressions obtained by a cyclic rotation in the indices l, m and n of eqn. (2.16) and using the following results (Rund 1959) :

$$p_j\phi_{kh} + p_k\phi_{hj} + p_h\phi_{jk} = 0,\quad \nu_j\phi_{kh} + \nu_k\phi_{hj} + \nu_h\phi_{jk} = 0$$

and the fact that $-\dot{\partial}_r\phi_{kh}X^r_t\,\dot{x}^t = \dot{\partial}_r X^r_t\,\phi_{kh}\dot{x}^t$, we get

$$\{d_{[lm(n)]} + \mu_{[l(m)(n)]} + d_{[lm}\nu_{n]} + \mu_{[lm}\nu_{n]}$$
$$+ \nu_{[l}\,U_{m(n)]} + \mu_{[l}\,U_{n(m)]} + d_{ln}\,U_{m]} + (d_{[ln}\,\mu_{m]}$$
$$+ \mu_{[l(n)}\,\mu_{m]}\}\,\phi_{kh} = (\beta_l\phi_{mn} + \beta_m\phi_{nl} + \beta_n\phi_{lm})\,\phi_{kh} \qquad \ldots(2.17)$$

where β_l is given by eqn. (2.14*b*).

Equation (2.17) completes the proof of Theorem 2.3.

Theorem 2.4 — In 2-RF_n, the decomposed tensor field ϕ_{kh} satisfies the relation :

$$6\phi_{kh[(l)(m)(n)]} = (\beta_l\phi_{mn} + \beta_m\phi_{nl} + \beta_n\phi_{lm})\,\phi_{kh}. \qquad \ldots(2.18)$$

PROOF : Taking cyclic permutation of eqn. (2.15), with respect to indices l, m, n and adding all the three equations thus obtained and using (2.17), we get the required result.

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