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BILINEAR GENERATING FUNCTIONS FOR ORTHOGONAL POLYNOMIALS RELATED TO ULTRASPHERICAL POLYNOMIALS

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This paper is the continuation of the study of the orthogonal polynomial solutions of the self-adjoint differential equations

$$D\,[(1 - x^{2k})^{\alpha+1}\,Dy] + \lambda(1 - x^{2k})^{\alpha}\,x^{2k-2}y = 0,$$

$$D \equiv d/dx;\ \alpha > -1,\ k = 1, 2, \ldots$$

initiated by Thakare and Bhonsle (1974). Here we obtain the closed expressions for the series

$$\sum_{p=0}^{\infty} T_{2pk}(u)\,T_{2pk}(v)\,\frac{t^p}{p!}$$

and

$$\sum_{p=0}^{\infty} T_{2pk+1}(u)\,T_{2pk+1}(v)\,\frac{t^p}{p!}$$

whereby the corresponding results for ultraspherical polynomials, Tchebycheff polynomials and Legendre polynomials are rendered intuitive.

§1. Recently, Thakare and Bhonsle (1974) have considered a new class of generalized orthogonal polynomials which are related to ultraspherical polynomials. They studied the polynomial solutions of the following self-adjoint differential equations

$$D\,[(1 - x^{2k})^{\alpha+1}\,Dy] + \lambda(1 - x^{2k})^{\alpha}\,x^{2k-2}y = 0 \qquad \ldots(1.1)$$

$$D \equiv d/dx;\ \alpha > -1, k = 1, 2, \ldots. \qquad \ldots(1.2)$$

The problem has been formulated in such a fashion that the above solutions of the self-adjoint differential equations constitute the ultraspherical polynomials as the first set (i.e. when $k = 1$). In fact this approach leads to an infinite number of complete orthogonal sets, each of which is composed in turn of an infinite number of polynomials. Thakare and Bhonsle showed that for eigenvalues $\lambda = n(n + 2k\alpha + 2k - 1)$ with $n \in \{2pk, 2pk + 1\}$, $(p = 0, 1, 2, \ldots)$, the solutions of (1.1) are even and odd orthogonal polynomials respectively with respect to the weight function $(1 - x^{2k})^{\alpha}$

x^{2k-2} over the interval $(-1, 1)$ and they denoted them by $T_n^{(\alpha,k)}(x)$, $n \in \{2pk, 2pk+1\}$ (see also Thakare and Karande 1973a and Karande and Thakare 1975).

The object of this paper is to obtain bilinear generating functions for the even and odd generalized orthogonal polynomials related to ultraspherical polynomials.

§2. Thakare and Bhonsle (1974) have given the Rodrigues' formulae for the even and odd generalised orthogonal polynomials respectively as

$$T_{2pk}^{(\alpha,k)}(x) = T_{2pk}(u) = \frac{(-1)^p u^\beta}{(1+\alpha)_p}(1-u)^{-\alpha}\frac{d^p}{du^p}[u^{p-\beta}(1-u)^{\alpha+p}] \quad \ldots(2.1)$$

and

$$T_{2pk+1}^{(\alpha,k)}(x) = T_{2pk+1}(u) = \frac{(-1)^p(1-u)^{-\alpha}}{(1+\alpha)_p}\frac{d^p}{du^p}[u^{p+\beta}(1-u)^{p+\alpha}] \quad \ldots(2.2)$$

where $\quad u = x^{2k}$ and $\beta = \dfrac{1}{2k}\cdot$

In view of (2.1) it would be of interest to find a bilinear generating function for the even generalized orthogonal polynomials $T_{2pk}^{(\alpha,k)}$. We examine the sum

$$\phi(t) = \sum_{p=0}^{\infty} T_{2pk}(u)\, T_{2pk}(v)\frac{t^p}{p!}\cdot \quad \ldots(2.3)$$

Making use of (2.1) it is evident that

$$\phi(t) = \sum_{p=0}^{\infty}\frac{t^p}{p!}\frac{(-1)^p u^\beta(1-u)^{-\alpha}}{(1+\alpha)_p}(D)^p u^{p-\beta}(1-u)^{\alpha+p}$$

$$\times\frac{(-1)^p v^\beta}{(1+\alpha)_p}(1-v)^{-\alpha}(D)^p v^{p-\beta}(1-v)^{\alpha+p}$$

$$= (1-u)^{-\alpha}(1-v)^{-\alpha}\sum_{r,s=0}^{\infty}\frac{(-\alpha-p)_r(-\alpha-p)_s u^r v^s}{r!\,s!}$$

$$\times\sum_{p=0}^{\infty}\frac{t^p}{p!\,(1+\alpha)_p(1+\alpha)_p}(-p+\beta-r)_p(-p+\beta-s)_p.$$

We have therefore proved that

$$\sum_{p=0}^{\infty} T_{2pk}(u)\, T_{2pk}(v)\frac{t^p}{p!} = (1-u)^{-\alpha}(1-v)^{-\alpha}.$$

$$\sum_{r,s=0}^{\infty} \frac{(-\alpha-p)_r\,(-\alpha-p)_s\,u^r v^s}{r!\,s!}$$

$$\times\ {}_2F_2\begin{bmatrix} -p+\beta-r, & -p+\beta-s & \\ & & ;t \\ 1+\alpha & , \quad 1+\alpha & \end{bmatrix}. \qquad \ldots(2.4)$$

We shall now derive bilinear generating function involving the odd generalized orthogonal polynomials on the basis of the simple nth differential formula given in (2.2). Exactly as in the proof of (2.4) we can show that

$$\sum_{p=0}^{\infty} T_{2pk+1}(u)\,T_{2pk+1}(v)\,\frac{t^p}{p!} = u^{\beta}v^{\beta}(1-u)^{-\alpha}\,(1-v)^{-\alpha}$$

$$\times \sum_{r,s=0}^{\infty} \frac{(-\alpha-p)_r\,(-\alpha-p)_s\,u^r v^s}{r!\,s!}$$

$$\times\ {}_2F_2\begin{bmatrix} -p-\beta-r, & -p-\beta-s & \\ & & ;t \\ 1+\alpha & , \quad 1+\alpha & \end{bmatrix}. \qquad \ldots(2.5)$$

Some special cases of generating functions (2.4) and (2.5) are worthy of note. For instance, if we put $\alpha = 0$, $k = 1$, we get

$$\sum_{p=0}^{\infty} P_{2p}(u)\,P_{2p}(v)\,\frac{t^p}{p!} = \sum_{r,s=0}^{\infty} \frac{(-p)_r\,(-p)_s\,u^r v^s}{r!\,s!}$$

$$\times\ {}_2F_2\begin{bmatrix} -p-r+\tfrac{1}{2}, & -p-s+\tfrac{1}{2} & \\ & & ;t \\ 1 & , \quad 1 & \end{bmatrix} \qquad \ldots(2.6)$$

and

$$\sum_{p=0}^{\infty} P_{2p+1}(u)\,P_{2p+1}(v)\,\frac{t^p}{p!} = (uv)^{1/2} \sum_{r,s=0}^{\infty} \frac{(-p)_r\,(-p)_s\,u^r v^s}{r!\,s!}.$$

$$\times\ {}_2F_2\begin{bmatrix} -p-r-\tfrac{1}{2}, & -p-s-\tfrac{1}{2} & \\ & & ;t \\ 1 & , \quad 1 & \end{bmatrix} \qquad \ldots(2.7)$$

Results (2.6) and (2.7) are unnoticed results for even and odd Legendre polynomials. Yet another interesting special case of the generating functions (2.4) and (2.5) for ultraspherical polynomials would occur when we set $k = 1$ and $\alpha > -1$.

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