Indian J. pure appl. Math., **10**(3) : 334-338, March 1979

ON $| C, 1 |_k$ SUMMABILITY FACTORS OF FOURIER SERIES

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(*Received* 14 *June* 1978; *after revision* 8 *November* 1978)

Concerning the absolute summability factors of Fourier-Lebesgue series, Hasiang has proved a theorem imposing additional restriction on $\phi(t)$ to obtain $| C, 1 |$-summability of $\Sigma A_n \lambda_n$. In this paper, the theorem of Hasiang for $| C, 1 |_k$-summability for a more general class of functions, is generalized.

§1. Let $f(x)$ be a function which is L-integrable and periodic with period 2π and suppose,

$$f(x) \sim \tfrac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \equiv \Sigma A_n(x)$$

and

$$\sum_{n=1}^{\infty} (b_n \cos nx - a_n \sin nx) \equiv \sum_{n=1}^{\infty} B_n(x)$$

we write

$$\phi(t) = \phi_x(t) = \tfrac{1}{2} \{f(x+t) + f(x-t) - 2f(x)\}$$

$$\psi(t) = \psi_x(t) = f(x+t) - f(x-t).$$

Let Σa_n be a given infinite series with partial sums s_n and let $t_n = t_n^0 = na_n$. By σ_n^α and t_n^α we denote the nth Cesàro means of order $\alpha(\alpha > -1)$ of the sequence $\{s_n\}$ and $\{t_n\}$ respectively.

Definition — The series Σa_n is said to be absolutely summable (C, α) with index k, or simply $| C, \alpha |_k$ $(k \geqslant 1)$, if

$$\Sigma n^{k-1} | \sigma_n^\alpha - \sigma_{n-1}^\alpha |^k < \infty. \qquad ...(1.1)$$

Summability $| C, \alpha |_1$ is the same as summability $| C, \alpha |$, since

$$t_n^\alpha = n(\sigma_n^\alpha - \sigma_{n-1}^\alpha).$$

Condition (1.1) can be replaced by

$$\sum \frac{| t_n^\alpha |^k}{n} < \infty.$$

Hasiang (1964) has proved the following theorem :

Theorem A — If

$$\Phi(t) = \int_0^t |\phi(u)|\, du = O\left(\frac{t}{\log \frac{1}{t}}\right), t \to 0$$

then the series

$$\sum_{n=2}^{\infty} \frac{A_n(x)}{(\log n)^{1+\epsilon}}$$

is summable $\| C, 1 \|$ for every $\epsilon > 0$.

Our object is to generalize the above result in the following two forms :

Theorem 1 — If

$$\Phi(t) = \int_0^t \| \phi(u) \|\, du = O\left(\frac{t}{\left(\log \frac{1}{t}\right)^{1/k}}\right), \text{ as } t \to \infty$$

$k \geqslant 1$, then the series

$$\sum_{n=2}^{\infty} \frac{A_n(x)}{(\log n)^{1+\epsilon}}$$

is summable $\| C, 1 \|_k$ for every $\epsilon > 0$.

Theorem 2 — Let $\chi(t)$ be a positive monotonic increasing function of t such that

(i) $\chi(t) = O(1), t \to 0,$

(ii) $\chi(t) \log\left(\frac{1}{t}\right) \to \infty$ as $t \to \infty$,

(iii) $t\chi'(t) \log\left(\frac{1}{t}\right) = O(\chi(t))$

and

(iv) $\sum_{n=1}^{\infty} \frac{1}{n} \left\{\frac{\chi(1/n)}{(\log n)^{\delta}}\right\}^k < \infty, k \geqslant 1$ for some $\delta > 0$.

If $$\Phi(t) = \int_0^t |\phi(u)|\, du = O(t(t\chi)) \text{ as } t \to 0$$

then $$\sum_{n=2}^{\infty} \frac{A_n(x)}{(\log n)^{1+\epsilon}}$$

is summable $\| C, 1 \|_k$ for every $\epsilon > 0$.

§2. *Proof of Theorem* 1 — Let $T_n(x)$ denote the nth Cesàro mean of first order of the sequence $\left\{\frac{nA_n(x)}{(\log n)^{1+\epsilon}}\right\}$, then

$$A_n(x) = \frac{2}{\pi} \int_0^{\pi} \phi(t) \cos nt\, dt.$$

We have

$$\frac{\pi}{2} T_n(x) = \int_0^{\pi} \phi(t) \frac{1}{n+1} \sum_{\nu=0}^{n} \frac{(\nu + 2) \cos (\nu + 2)\, t}{(\log (\nu + 2))^{1+\epsilon}}\, dt$$

so that

$$\frac{\pi}{2} | T_n(x) | \leqslant A \int_0^{1/n} | \phi(t) | \frac{1}{n+1} \frac{n}{(\log n)^{1+\epsilon}} \sum_{\nu=0}^{n} | \cos (\nu + 2) t) |\, dt$$

$$+ A \int_{1/n}^{\pi} | \phi(t) | \frac{1}{n+1} \frac{n}{(\log n)^{1+\epsilon}} \operatorname*{Max}_{0 \leqslant m \leqslant n} \left| \sum_{\nu=0}^{m} \cos (\nu + 2)\, t \right| dt$$

$$= O\left\{\frac{n}{(\log n)^{1+\epsilon}} \int_0^{1/n} | \phi(t) |\, dt\right\} + O\left(\frac{1}{(\log n)^{1+\epsilon}} \int_{1/n}^{\pi} \frac{| \phi(t) |}{t}\, dt\right)$$

$$= I_1 + I_2 \text{ (say)}.$$

Now $$I_1 = O\left[\frac{n}{(\log n)^{1+\epsilon}} \left(\frac{1}{n} \frac{1}{(\log n)^{1/k}}\right)\right]$$

$$= O\left(\frac{1}{(\log n)^{1+\epsilon+(1/k)}}\right).$$

Also by integration by parts and the hypothesis of the theorem,

$$\int_{1/n}^{\pi} \frac{| \phi(t) |}{t}\, dt = O(\log n)^{(k-1)/k}.$$

Hence

$$I_2 = O(\log n)^{-(1/k)-\varepsilon}.$$

Thus

$$T_n(x) = O(\log n)^{-(1/k)-\varepsilon}$$

so that

$$\sum_{n=2}^{\infty} \frac{|T_n(x)|^k}{n} = O\left\{\sum_{n=2}^{\infty} \frac{(\log n)^{-(k\varepsilon+1)}}{n}\right\}$$

$$= O(1).$$

This completes the proof of the Theorem 1.

§3. *Proof of Theorem* 2 — Following the argument given in the proof of Theorem 1, we have

$$I_1 = O\left[n(\log n)^{-1-\varepsilon} \int_0^{1/n} |\phi(t)|\, dt\right]$$

$$= O\left[\frac{\left(\frac{1}{n}\right)}{(\log n)^{1+\varepsilon}}\right]$$

$$I_2 = \left\{\frac{1}{(\log n)^{1+\varepsilon}} \int_{1/n}^{\pi} \frac{|\phi(t)|}{t}\, dt\right\}.$$

Now, by integration by parts and using the hypothesis of Theorem 2, we have

$$I_2 = O\left(\frac{1}{(\log n)^{1+\varepsilon}} \log n \chi\left(\frac{1}{n}\right)\right).$$

$$= O\left(\frac{\chi\left(\frac{1}{n}\right)}{(\log n)^{\varepsilon}}\right).$$

Thus,

$$T_n(x) = O\left\{(\log n)^{-\varepsilon} \chi\left(\frac{1}{n}\right)\right\},$$

$$\sum_{n=2}^{\infty} \frac{|T_n(x)|^k}{n} = O\left[\sum_{n=2}^{\infty} \frac{1}{n} \left(\frac{\chi(1/n)}{(\log n)^{\varepsilon}}\right)^k\right] = O(1),$$

choosing $\delta \leqslant \epsilon$.

This completes the proof of Theorem 2.

§4. For the conjugate Fourier series we can prove analogous theorems.

Theorem 3 — If

$$\Psi(t) = \int_0^t |\psi(u)|\,du = O\left(\frac{t}{\left(\log \frac{1}{t}\right)^{1/k}}\right) \text{ as } t \to 0,\ k \geqslant 1$$

then the series

$$\sum_{n=2}^{\infty} \frac{B_n(x)}{(\log n)^{1+\epsilon}},\ \epsilon > 0 \text{ is summable } |C, 1|_k.$$

Theorem 4 — Let $\chi(t)$ be a positive monotonic increasing function of t such that

(i) $\chi(t) = O(1),\ t \to 0,$

(ii) $\chi(t) \log \frac{1}{t} \to \infty$ as $t \to 0,$

(iii) $t\chi'(t) \log \frac{1}{t} = O(\chi(t)),$

and (iv) $\sum_{n=1}^{\infty} \frac{1}{n} \left\{\frac{\chi(1/n)}{(\log n)^{\delta}}\right\}^k < \infty,\ k \geqslant 1$ for some $\delta > 0$

If $\Psi(t) = \int_0^t |\psi(u)|\,du = O\{t\,.\,\chi(t)\},\ t \to 0,$ then

$$\sum_{n=2}^{\infty} \frac{B_n(x)}{(\log n)^{1+\epsilon}},\ \epsilon > 0$$

is summable $|C, 1|_k$.

Reference

Hasiang (1964). On the absolute summability factors of series at a point. *Comp. Math.*, **17**, 156-60.