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A CLASS OF NON-STATIC CYLINDRICALLY SYMMETRIC COUPLED ZERO-MASS AND ELECTROMAGNETIC FIELDS—I

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A class of relations, may be called as Majumdar-Papapetrou class, obtained by Misra and Radhakrishna (1962) for non-static cylindrically symmetric source-free Einstein-Maxwell fields have been studied to the case of coupled source-free electromagnetic and zero-mass fields. It has been found that there exist certain physical situations (in a sense quite general) in which the presence of zero-mass field does not affect the said MP relations. We have also given, for the case under consideration, a theorem to generate a class of solutions, which in a way can be said to belong to MP class, from the non-static cylindrically symmetric zero-mass fields.

1. INTRODUCTION

Majumdar (1947) and Papapetrou (1947) have, independently, shown that for the generalized static metric

$$ds^2 = g_{ab}dx^a dx^b + g_{44}dt^2 (a, b = 1, 2, 3 \text{ and } g_{44} < 0) \quad ...(1)$$

the (4-4) component of the metric potential is related to the electrostatic potential, ψ, as

$$-g_{44} = \frac{\kappa}{8\pi}(A + B\psi + \psi^2) \quad ...(2)$$

where A and B are arbitrary constants. The class of solutions generated by (2) has been the subject of extensive investigation by various authors (Perjés 1971, Israel and Wilson 1972, Hartle and Hawking 1972). Misra and Radhakrishna (1962), while obtaining the exact solutions for non-static cylindrically symmetric source-free Einstein-Maxwell fields described by Einstein-Rosen metric, have incidently obtained analogous relations between some components of the metric and electromagnetic potentials. The relations obtained by them are :

$$g_{22} = \frac{\kappa}{8\pi}(-a - b\psi + \psi^2) \quad ...(3)$$

and

$$g_{33} = \frac{\kappa}{8\pi}(-a - b\phi + \phi^2) \quad ...(4)$$

where ψ and ϕ are A_3 and A_2 components of the 4-potential, A_μ. Later Rao *et al.* (1972) extended the problem of Misra and Radhakrishna (1962) to the case of cylindrically symmetric coupled electromagnetic and zero-mass fields and obtained a number of exact solutions. They have, however, ignored to observe the effect of zero-mass field on the relations (3) and (4). In the present paper we have investigated this aspect of the problem.

Various physical situations, depending on whether the gradient vectors of the electromagnetic and scalar field potentials are null, non-null or mutually orthogonal, may arise. The case studied here, however, in a sense is more general as it does not involve any such restriction. In another sense it is less general since to avoid any restriction on the nature of physical fields (such as mentioned above) the mathematical constraints are to be increased. Consequently the consistency requirement of these equations has narrowed down the domain of validity of the MP relations and thereby MP class of solutions. The other physical cases, therefore, naturally deserve serious consideration which we propose to study in our future work.

We have shown that, in general (in the sense discussed here), the presence of zero-mass field does not modify the MP relations (3) or (4). In fact, if the metric potential g_{22} (or g_{33}) is assumed to be dependent on the electromagnetic as well as zero-mass fields the differential equations pertaining to MP relations lead to the imaginary value of these potentials. Consequently the metric potential g_{22} (or g_{33}) depends either only on the electromagnetic potential [in which case MP relations are the same as those obtained by Misra and Radhakrishna (1962)] or only on the zero mass field. The linear coupling of zero-mass does not affect the MP relations (3) or (4).

In Section 2, we have set the relevant material for the problem. Section 3 deals with the field equations and obtaining the MP relations which has further been used to generate a class of solutions, the so called MP class, from zero-mass non-static cylindrically symmetric fields. It may be mentioned that the astrophysical role of the static and the stationary MP class of solutions has been discussed by Perjés (1971), Israel and Wilson (1972) and Hartle and Hawking (1972).

2. Field Equations

The general relativistic field equations, in the presence of coupled electromagnetic and zero rest mass scalar fields, may be written as

$$R_{ij} - \tfrac{1}{2} R g_{ij} = - \kappa (T^{(e)}_{ij} + T^{(s)}_{ij}) \qquad \ldots(5)$$

where $T^{(e)}_{ij}$ and $T^{(s)}_{ij}$ correspond to the source-free electromagnetic and scalar fields, respectively, and are given as

$$T^{(e)}_{ij} = \frac{1}{4\pi}\left(-F_{is}F^{s}_{j} + \tfrac{1}{4}g_{ij}F_{sp}F^{sp}\right) \qquad \ldots(6)$$

and

$$T^{(s)}_{ij} = \frac{1}{4\pi}\left(v_{,i}v_{,j} - \tfrac{1}{2}g_{ij}v_{p}v^{p}\right) \qquad \ldots(7)$$

Here, F_{ij}, the components of the electromagnetic field tensor and, v, the scalar field satisfy the relations,

$$F_{ij} = A_{i,j} - A_{j,i} \qquad \ldots(8)$$

$$F^{ij}_{;j} = 0 \qquad \ldots(9)$$

$$g^{ij}v_{;ij} = 0 \qquad \ldots(10)$$

where a comma and a semicolon denote partial and covariant differentiations respectively.

We now consider the non-static cylindrically symmetric Einstein-Rosen metric

$$ds^2 = e^{2\alpha-2\beta}(dt^2 - d\rho^2) - \rho^2 e^{-2\beta}d\Phi^2 - e^{2\beta}dz^2. \qquad \ldots(11)$$

We adopt the convention of denoting the coordinates ρ, Φ, z, t as x^1, x^2, x^3, x^4 respectively. It may be verified, as obtained in Rao *et al.* (1972), the components $V_{,2}$, $V_{,3}$, $A_{i,2}$, $A_{i,3}$, F_{14} and F_{23} are identically zero by virtue of the symmetry imposed by (11). The remaining components of F_{ij} can be expressed only in terms of two independent potentials, A_2 and A_3, of the 4-electromagnetic potential, A_i. We denote, $A_2 = \phi$ and $A_3 = \psi$, and hence, we get

$$F_{12} = -\phi_1,\ F_{13} = -\psi_1,\ F_{24} = \phi_4,\ F_{34} = \psi_4.$$

Here lower suffixes 1 and 4 after unknowns denote partial differentiations with respect to ρ and t, respectively. In view of the above, the field eqns. (5) reduce to the following :

$$\beta_4^2 + \beta_1^2 - \frac{\alpha_1}{\rho} = -\frac{\kappa}{8\pi}\left\{\frac{e^{2\beta}}{\rho^2}(\phi_1^2 + \phi_4^2) + e^{-2\beta}(\psi_1^2 + \psi_4^2)\right\} - \frac{\kappa}{8\pi}(v_1^2 + v_4^2) \qquad \ldots(12)$$

$$\beta_{11} - \beta_{44} + \frac{\beta_1}{\rho} = \frac{\kappa}{8\pi}\left\{\frac{e^{2\beta}}{\rho^2}(\phi_1^2 - \phi_4^2) - e^{-2\beta}(\psi_1^2 - \psi_4^2)\right\} \qquad \ldots(13)$$

$$\frac{\alpha_4}{\rho} - 2\beta_1\beta_4 = \frac{\kappa}{8\pi}\left(\frac{e^{2\beta}}{\rho^2}\phi_1\phi_4 + e^{-2\beta}\psi_1\psi_4\right) + \frac{\kappa}{8\pi}v_1v_4 \qquad \ldots(14)$$

$$R_3^2 \equiv \phi_1\psi_1 - \phi_4\psi_4 = 0. \qquad \ldots(15)$$

Also from (9) and (10), we have

$$\phi_{11} - \phi_{44} - \frac{\phi_1}{\rho} = 2\beta_4\phi_4 - 2\beta_1\phi_1 \qquad \text{...(16)}$$

$$\psi_{11} - \psi_{44} + \frac{\psi_1}{\rho} = 2\beta_1\psi_1 - 2\beta_4\psi_4 \qquad \text{...(17)}$$

$$\nu_{11} - \nu_{44} + \frac{\nu_1}{\rho} = 0. \qquad \text{...(18)}$$

We now make the following two assumptions, viz.,

(a) $\phi = 0,\ \psi \neq 0,\ \nu \neq 0,$

(b) $\phi \neq 0,\ \psi = 0,\ \nu \neq 0.$

3. MP Relations for Coupled Field

We shall now obtain MP relations for both the above cases. In fact, we will obtain the relation for the first case and then since the procedure is exactly the same for the second case also the corresponding result will simply be mentioned at the end. The field equations in the first case, viz. $\phi = 0,\ \psi \neq 0,\ \nu \neq 0$, reduce to

$$\beta_{11} - \beta_{44} + \frac{\beta_1}{\rho} = -\ e^{-2\beta}(\psi_1^2 - \psi_4^2)\,\frac{\kappa}{8\pi} \qquad \text{...(19)}$$

$$\psi_{11} - \psi_{44} + \frac{\psi_1}{\rho} = 2\beta_1\psi_1 - 2\beta_4\psi_4 \qquad \text{...(20)}$$

$$\alpha_1 = \rho(\beta_1^2 + \beta_4^2) + \frac{\kappa\rho}{8\pi}\,e^{-2\beta}(\psi_1^2 + \psi_4^2) + \frac{\kappa\rho}{8\pi}\,(\nu_1^2 + \nu_4^2) \qquad \text{...(21)}$$

$$\alpha_4 = 2\rho\beta_1\beta_4 + \frac{\kappa\rho}{4\pi}\,e^{-2\beta}\psi_1\psi_4 + \frac{\kappa\rho}{4\pi}\,\nu_1\nu_4 \qquad \text{...(22)}$$

$$\nu_{11} - \nu_{44} + \frac{\nu_1}{\rho} = 0. \qquad \text{...(23)}$$

We assume β to be dependent on ψ and ν both, viz.

$$\beta = \beta(\psi, \nu). \qquad \text{...(24)}$$

In view of (24), the eqn. (20) becomes

$$\psi_{11} - \psi_{44} + \frac{\psi_1}{\rho} = 2\beta_\psi(\psi_1^2 - \psi_4^2) + 2\beta_\nu(\nu_1\psi_1 - \nu_4\psi_4). \qquad \text{...(25)}$$

Using (23), (24) and (25) in (19), after simplification, we get

$$\left(\beta_{\psi\psi} + 2\beta_{\psi}^2 + \frac{\kappa}{8\pi} e^{-2\beta}\right)(\psi_1^2 - \psi_4^2) + 2(\beta_{\psi v} + \beta_\psi\beta_v)(v_1\psi_1 - v_4\psi_4)$$
$$+ \beta_{vv}(v_1^2 - v_4^2) = 0. \qquad \ldots(26)$$

Here ψ and v are independent of each other. Again the terms* $(\psi_1^2 - \psi_4^2)$, $(v_1^2 - v_4^2)$, $(v_1\psi_1 - v_4\psi_4)$ can all be zero only when the gradient vectors of the electromagnetic potential, ψ, and the zero-mass field, v, (hereafter electromagnetic and zero mass forcefields) are both null and mutually orthogonal. In such a situation eqn. (26) is identically satisfied and there may or may not exist any MP relation. We would, however, like to avoid such a restricted physical situation to include the non-null and non-orthogonal force fields also. The other particular cases will be studied in our future work. Thus equating the coefficients of $(\psi_1^2 - \psi_4^2)$, $(v_1^2 - v_4^2)$ and $(v_1\psi_1 - v_4\psi_4)$ equal to zero, so that eqn. (26) is identically satisfied, we get the following differential conditions :

$$\beta_{\psi\psi} + 2\beta_{\psi}^2 + \frac{\kappa}{8\pi} e^{-2\beta} = 0 \qquad \ldots(27)$$

$$\beta_{\psi v} + \beta_\psi\beta_v = 0 \qquad \ldots(28)$$

$$\beta_{vv} = 0 \qquad \ldots(29)$$

where $\beta_\psi = \dfrac{\partial\beta}{\partial\psi}$, $\beta_v = \dfrac{\partial\beta}{\partial v}$ etc.

It may be remarked that these conditions may hold even if the electromagnetic and zero-mass force-fields are null and mutually orthogonal.

Integrating (27), we get

$$-g_{33} = e^{2\beta} = \frac{\kappa}{8\pi}\{a(v) + b(v)\psi - \psi^2\}. \qquad \ldots(30)$$

Similarly integrating (28), we have

$$e^{\beta} = A(v) + B(\psi). \qquad \ldots(31)$$

Equation (31) demands that (30) must be a perfect square. Comparing (30) with (31), we find that

$$e^{\beta} = i\left(\psi + \frac{2}{b}\right) \qquad \ldots(32)$$

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*The terms $e^{2\alpha-2\beta}(\psi_1^2 - \psi_4^2)$, $e^{2\alpha-2\beta}(v_1^2 - v_4^2)$, $e^{2\alpha-2\beta}(v_1\psi_1 - v_4\psi_4)$ are the values of $g^{ij}\psi_{,i}\psi_{,j}$, $g^{ij}v_{,i}v_{,j}$ and $g^{ij}v_{,i}\psi_{,j}$, respectively, for the metric (11).

which gives an imaginary β. Hence dependence of β on ψ and v, both amounts to its imaginary value. On the other hand β depending on ψ alone gives the usual relation obtained by Misra and Radhakrishna (1962). Thus one can conclude that, β, in order to be real, must depend on ψ alone. This restricts the solutions generated by MP class of relations only to those which could be obtained by using the MP relation of Misra and Radhakrishna (1962), viz.,

$$-g_{33} = e^{2\beta} = \frac{\kappa}{8\pi}(a + b\psi - \psi^2) \qquad ...(30a)$$

where a and b are arbitrary constants.

Using first the relation between the metric and the electromagnetic potentials given by (3) and then substituting

$$\int \frac{d\psi}{a + b\psi - \psi^2} = \frac{2\gamma}{\sqrt{4a + b^2}} \qquad ...(33)$$

the field eqns. (19) to (23) reduce to

$$\gamma_{11} - \gamma_{44} + \frac{\gamma_1}{\rho} = 0 \qquad ...(34)$$

$$\alpha_1 = \rho(\gamma_1^2 + \gamma_4^2) + \frac{\kappa\rho}{8\pi}(v_1^2 + v_4^2) \qquad ...(35)$$

$$\alpha_4 = 2\rho\gamma_1\gamma_4 + \frac{\kappa\rho}{4\pi} v_1 v_4 \qquad ...(36)$$

$$v_{11} - v_{44} + \frac{v_1}{\rho} = 0. \qquad ...(37)$$

Also the eqn. (33) integrates as

$$\psi = \tfrac{1}{2} b + \tfrac{1}{2}\sqrt{4a + b^2}\tanh\gamma. \qquad ...(38)$$

Using this in (30*a*), we get

$$\beta = A + \log\operatorname{sech}\gamma \qquad ...(39)$$

where $$A = \tfrac{1}{2}\log\frac{\kappa}{8\pi}\left(\frac{b^2 + 4a}{4}\right).$$

It may be verified that eqns. (34) – (37) correspond to those of zero-mass field for the cylindrically symmetric metric

$$ds^2 + e^{2\alpha-2\gamma}(dt^2 - d\rho^2) - \rho^2 e^{-2\gamma} d\Phi^2 - e^{2\gamma} dz^2. \qquad ...(40)$$

Thus given any solution of cylindrically symmetric zero mass field described by the metric (40) one can construct the corresponding coupled field with the help of relations (38) and (39). The result may be stated as follows :

Theorem — Given any cylindrically symmetric zero mass solution $(\alpha_0, \gamma_0, \nu_0)$ for the metric (40), it is always possible to generate a corresponding coupled electromagnetic and zero mass field solution $(\alpha, \gamma, \nu, \psi)$, where

$$\alpha = \alpha_0,\ \gamma = A + \log \operatorname{sech} \gamma_0,\ \nu = \nu_0$$

and

$$\psi = \tfrac{1}{2} b + \tfrac{1}{2} \sqrt{4a + b^2} \tanh \gamma_0. \qquad \ldots(41)$$

As an application of the above theorem, an important solution is obtained by using the Bessel function solution, of the set of eqns. (34) - (37), given by

$$\gamma_0 = BJ_0(\omega\rho) \cos \omega t \qquad \ldots(42)$$

$$\nu_0 = CJ_0(\omega\rho) \sin \omega t \qquad \ldots(43)$$

and

$$\alpha_0 = \tfrac{1}{2} \left(B^2 + \frac{\kappa C^2}{8\pi}\right) \omega\rho J_0(\omega\rho)\, J_0'(\omega\rho) \cos 2\omega t$$

$$+ \tfrac{1}{2} \left(B^2 + \frac{\kappa C^2}{8\pi}\right) \omega^2\rho^2 \{[J_0'(\omega\rho)]^2 - J_0(\omega\rho)\, J_0''(\omega\rho)\}. \qquad \ldots(44)$$

The required solution is obtained by substituting (42) - (44) in (41). One may refer to Rosen (1954) for the physical interpretation of solutions (42) - (44).

Remark 1 : As mentioned in section (3), in the second case (viz. $\phi \neq 0$, $\psi = 0$, $\nu \neq 0$) also the MP relation remains unaffected by the addition of zero mass field and is given by eqn. (4).

Remark 2 : In view of the above a natural question arises whether there is any effect of zero mass field on the MP relation (2) in the case of static cylindrically symmetric metric. We may mention here that as usual, as obtained in Section 3, there is no such effect. As to the question whether zero mass field modifies the MP relation (2), for the generalized static metric (1), we are examining the detail and will report in our future work.

Remark 3 : The eqns. (34) - (37) can further be reduced to Einstein vacuum equations for the metric (40) by taking ν and γ to be proportional. This, thus amounts to generating the solutions of the set (34) - (37) from vacuum solutions. Indeed, the solutions (42) - (44) for zero mass field have been constructed by using this technique.

Remark 4 : The investigations carried out above are subject to restrictions discussed in Section 3. The physical possibilities in which cases the MP relations may be affected by the presence of zero mass field will be reported in our future work.

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