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VIBRATIONS AND BUCKLING OF PARABOLICALLY TAPERED CIRCULAR PLATES

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The natural frequencies of a circular plate of parabolically varying thickness resting on elastic foundation of Winkler type under the action of a hydrostatic inplane force are discussed on the basis of classical plate theory. The governing differential equation of motion is solved by the method of Frobenius'. Frequency parameters have been computed in the first two modes of vibration for clamped as well as simply supported plates for various values of taper parameter β, foundation moduli K and the inplane force $\bar{N}$. The effect of taper parameter β on the critical buckling load in compression, transverse displacements and moments has also been investigated.

1. INTRODUCTION

The study of buckling and vibrations of plates of non-uniform thickness are acquiring greater importance because of their applications in various fields such as aerospace engineering, civil engineering, machine design and the design of earthquake resistant structures, etc. As regards the literature an excellent survey of previous studies dealing with vibrating plates has been prepared by Leissa (1969), which reveals the fact that comparatively little work has been done on vibration problems of plates of variable thickness. Numerous studies reported in references (Conway 1958, 1959; Harris and Mansfield 1967; Harris 1968; Jain 1972; Jain and Soni 1973; Laura *et al.* 1977; Prasad *et al.* 1972; Tomar and Gupta 1976; Warburton 1975) on the free vibrations of plates of variable thickness indicate that no work has been done to investigate the combined effect of the inplane forces together with the elastic foundation on the natural frequencies, when the plate is of varying thickness.

The object of the present paper is to investigate the combined effect of elastic foundation together with that of thickness variation and the hydrostatic inplane force on the natural frequencies of a circular plate of parabolically varying thickness on the basis of classical theory. This analysis is valid only in the low frequency range for thin plates. The solution has been obtained by Frobenius' method i.e. a power series development of the transverse deflection function. Frequency parameters of clamped and simply supported plates in the first two modes of vibration have been computed for various values of taper constant β, inplane force $\bar{N}$ and the foundation moduli K. The effect of thickness variation on the critical buckling load in compression has also been investigated.

2. Equation of Motion

The small deflection axisymmetric motion of a thin circular plate of radius a, thickness $h(r)$, density ρ, Poisson's ratio ν resting on an elastic foundation of foundation modulus K_f is governed by the well-known equation (Gupta and Lal 1978)

$$\nabla^2(h^3 \nabla^2 w) - (1 - \nu) \frac{1}{r} \left(\frac{\partial^2 h^3}{\partial r^2} \frac{\partial w}{\partial r} + \frac{\partial^2 w}{\partial r^2} \frac{\partial h^3}{\partial r} \right)$$

$$+ \frac{12(1 - \nu^2)}{E} \left(K_f w + \rho h \frac{\partial^2 w}{\partial t^2} - N \nabla^2 w \right) = 0 \quad \ldots(1)$$

where t is the time, r the radial co-ordinate, $w(r, t)$ the transverse deflection, E Young's modulus of the plate material, N the uniform inplane tensile force and ∇^2 the two-dimensional Laplacian operator in polar coordinates.

The thickness variation can be of any type (Conway 1958, Jain and Soni 1973, Prasad *et al.* 1972, Warburton 1975). Quite a large number of papers of non-uniform thickness variation have come to the attention of the authors in which the variation of thickness is parabolic i.e.

$$\bar{h} = h_0(1 - \beta x^2)$$

where h_0 is thickness at the centre, β is the taper parameter and $\bar{h} = h/a$, $\bar{w} = w/a$, $x = r/a$ are the non-dimensional variables. The vibrations of circular plates of linearly varying thickness $\bar{h} = h_0(1 - \alpha x)$ have already been studied (Gupta and Lal 1978). Equation (1) will now be reduced to

$$(1 - \beta x^2)^3 \bar{w}_{,xxxx} + 2 [(1/x) - 9\beta x + 15\beta^2 x^3 - 7\beta^3 x^5] \bar{w}_{,xxx}$$

$$+ [- (1/x^2) + A_1 + 3\beta^2 A_2 x^2 + \beta^3 A_3 x^4] \bar{w}_{,xx}$$

$$+ [(1/x^3) + A_4(1/x) + 3\beta^2 A_5 x + 5\beta^3 A_6 x^3] \bar{w}_{,x}$$

$$+ \left[\frac{aK_f}{D_0} + \frac{\rho h_0 a^2}{D_0} (1 - \beta x^2) \frac{\partial}{\partial t^2} \right] \bar{w} = 0 \quad \ldots(2)$$

where

$$A_1 = 3\beta(-7 + 2 < 1 - \nu >) - (N/aD_0), \quad A_2 = (23 - 4 < 1 - \nu >),$$

$$A_3 = (-47 + 6 < 1 - \nu >), \quad A_4 = 3\beta(-1 + 2 < 1 - \nu >) - (N/aD_0),$$

$$A_5 = (9 - 12 < 1 - \nu >), \quad A_6 = (-5 + 6 < 1 - \nu >)$$

and $\quad D_0 = Eh_0^3/12(1 - \nu^2).$

3. Method of Solution

For harmonic vibrations

$$\bar{w}(x, t) = W(x)\, e^{i\omega t} \qquad \ldots(3)$$

where ω is the radian frequency of vibration. Let us assume a series solution for $W(x)$ in the form

$$W(x) = \sum_{\gamma=0}^{\infty} a_\gamma x^{C+\gamma} : a_0 \neq 0 \qquad \ldots(4)$$

where C is the exponent of singularity.

Substitution of (3) and (4) in (2) leads to the indicial equation

$$C^2(C-2)^2 = 0. \qquad \ldots(5)$$

Obviously $C = 0, 0, 2, 2$ are the roots of (5). Since the individual roots are repeated, two of the solutions will be singular at the origin and hence omitted. The constants a_0 and a_2 remain indeterminate while $a_1 = 0$ and a_γ ($\gamma = 3, 4, 5, \ldots$) are all determined from the following recurrence relation

$$\begin{aligned}(C+\gamma+4)^2 (C+\gamma+2)^2 a_{\gamma+4} + [A_4 + \langle C+\gamma+1 \rangle \\ \times (A_1 - 3\beta \langle C+\gamma \rangle \langle C+\gamma+5 \rangle)] (C+\gamma+2) \\ a_{\gamma+2} + [A_7 + 3\beta^2 \langle C+\gamma \rangle \{A_5 + \langle C+\gamma-1 \rangle \\ \times (A_2 + \langle C+\gamma-2 \rangle \langle C+\gamma+7 \rangle)\}] a_\gamma \\ + [\beta\Omega^2 + \beta^3 \langle C+\gamma-2 \rangle \{5A_6 + \langle C+\gamma-3 \rangle \\ \times (A_3 - \langle C+\gamma-4 \rangle \langle C+\gamma+9 \rangle)\}] a_{\gamma-2} = 0\end{aligned} \qquad \ldots(6)$$

where

$$A_7 = \left(\frac{aK_f}{D_0} - \Omega^2\right),\ \Omega^2 = \frac{12}{h_0^2}\, \Omega_p^2 \text{ and } \Omega_p^2 = \rho a^2 \omega^2 (1-\nu^2)/E.$$

It is found that $a_{2\gamma+1} = 0$ for $\gamma = 1, 2, 3, \ldots$. While the remaining coefficients involve a_0 and a_2. Let

$$a_\gamma = f_\gamma a_0 + g_\gamma a_2,\ \gamma = 0, 2, 4, 6, \ldots . \qquad \ldots(7)$$

It is easily seen that $f_0 = 1$, $g_0 = 0$, $f_2 = 0$, $g_2 = 1$ and f_γ, g_γ ($\gamma = 4, 6, 8, \ldots$) are functions of β, ν, $\bar{N}(= N/aD_0)$, $K\,(= K_f\, a/E)$ and Ω. The solution for $W(x)$, corresponding to $C = 0$ can be written as

$$W(x) = a_0 U(x, \Omega) + a_2 V(x, \Omega) \qquad \ldots(8)$$

where $U(x, \Omega) = 1 + \sum_{\gamma=4}^{\infty} f_\gamma x^\gamma$ and $V(x, \Omega) = \sum_{\gamma=2}^{\infty} g_\gamma x^\gamma$.

No new solution arises due to the indicial root $C = 2$ (it is already included in the solution corresponding to $C = 0$). The convergence of the solution (8) can be proved as in Jain (1972).

4. Boundary Conditions

The boundary conditions which have been considered here are:

(i) Clamped at the edge $x = 1$,

(ii) Simply supported at the edge $x = 1$.

The boundary conditions which should be satisfied at the clamped edge are

$$W = \frac{dW}{dx} = 0; \qquad \text{...(9)}$$

and at the simply supported edge

$$W = \left(\frac{d^2W}{dx^2} + \frac{\nu}{x}\frac{dW}{dx}\right) = 0. \qquad \text{...(10)}$$

Applying the boundary condition (9) to the displacement function (8) at the edge $x = 1$, one obtains the characteristic equation given by

$$\left[U(x, \Omega)\frac{dV(x, \Omega)}{dx} - V(x, \Omega)\frac{dU(x, \Omega)}{dx}\right]_{x=1} = 0. \qquad \text{...(11)}$$

Similarly for simply supported plate at the edge $x = 1$, the frequency equation is

$$\left[U(x, \Omega)\left\{\frac{d^2V(x, \Omega)}{dx^2} + \frac{\nu}{x}\frac{dV(x, \Omega)}{dx}\right\} - V(x, \Omega)\left\{\frac{d^2U(x, \Omega)}{dx^2} + \frac{\nu}{x}\frac{dU(x, \Omega)}{dx}\right\}\right]_{x=1} = 0 \qquad \text{...(12)}$$

respectively.

5. Deflection and Moments

The deflection and moments at different points of the plate (clamped or simply supported) are given by

$$W = U(x, \Omega) - \eta V(x, \Omega) \qquad \text{...(13)}$$

$$M_r = -D\left[U''(x, \Omega) + \frac{\nu}{x}U'(x, \Omega) - \eta\left\{V''(x, \Omega) + \frac{\nu}{x}V'(x, \Omega)\right\}\right] \qquad \text{...(14)}$$

where $\eta = U(1, \Omega)/V(1, \Omega)$ and primes denote the derivative with respect to x.

6. Discussion of Results

The characteristic eqns. (11) and (12) are transcendental equations in the frequency parameter Ω, which can be solved numerically for different values of the parameters $\bar{N}\left(=\frac{N}{aD_0}\right)$, $K\left(=\frac{K_f a}{E}\right)$ and β. The values of Ω have been computed for the first two axisymmetric modes of vibration both for clamped as well as simply supported circular plate for different values (+ ive and − ive) of the taper constant β. Effect of the taper constant β, the foundation modulus K, and that of the inplane force parameter $\bar{N}$ on the frequencies have been investigated for $\nu = 0.3$. The results agree with those of Jain (1972) when the effect of elastic foundation is neglected. The computations reported in this paper were carried out on IBM 360 computer.

For numerical results the series (8) has been computed to include all the terms up to an accuracy of 10^{-6} in their absolute value. It is found that the frequency parameter for a simply supported plate is less than the corresponding frequency parameter for a clamped plate. From Fig. 1, it is clear that for clamped plates vibrating in the fundamental mode, the frequency parameter decreases with increase in β and this decrease slows down with the increase in foundation modulus. The difference between the values of the frequency parameter (with and without elastic foundation) increases with the increase in taper parameter β, showing that the effect of elastic foundation becomes more and more pronounced as the plate gets thicker and thicker around the centre. In case of simply supported plates vibrating in the fundamental mode, the frequency parameter decreases with increase in β for small values of the foundation modulus parameter K, but the decrease is very slow. A

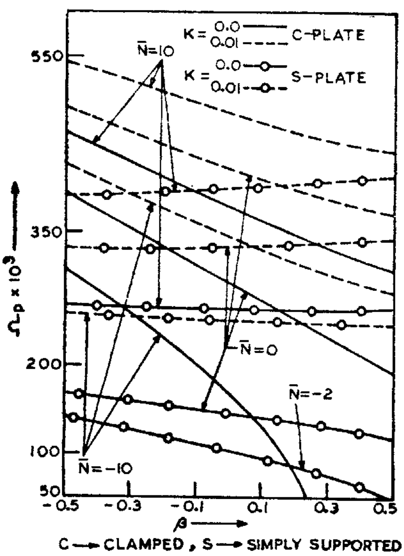

FIG. 1. Natural frequencies in the fundamental mode.

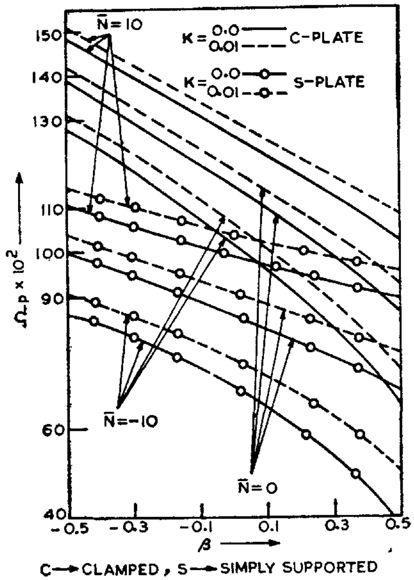

FIG. 2. Natural frequencies in the second mode.

marked difference is, however seen with the increase in foundation modulus. Fig. 1 shows that for $K = 0.01$ the frequency parameter increases with increase in β. The increase becomes more and more appreciable with both $\bar{N}$ and K increasing.

From Fig. 2, showing plots in the second mode of vibration, it is seen that the frequency parameter decreases continuously with the increase in taper constant β for both the plates clamped as well as simply supported. The presence of elastic foundation is found to increase the frequency parameter for both the boundary conditions. The difference between the values of frequency parameters (with and without elastic foundation) increases with the increase in β as in case of fundamental mode, but this difference is small as compared to the corresponding difference in the fundamental mode. It is interesting to note that for a given foundation modulus the percentage change in frequency in the fundamental mode is greater than in the case of second mode.

Normalized transverse displacements W_{norm} $(= W/W_{max})$ and moment $M_{norm}(= M/M_{max})$ in the first three modes of vibration are plotted in Figs. 3 and 4 for $\beta = \pm\, 0.3$, $K = 0.01$ and $\bar{N} = 2.0$ for both clamped and simply supported plates. The transverse deflection will be maximum at the centre for both the plates.

It is seen that the transverse deflection for $\beta = -0.3$ is less than the corresponding deflection for $\beta = 0.3$ in all the cases and the radii of nodal circles for $\beta = -0.3$ are less than those for $\beta = 0.3$. Physical explanation for this may be attributed to

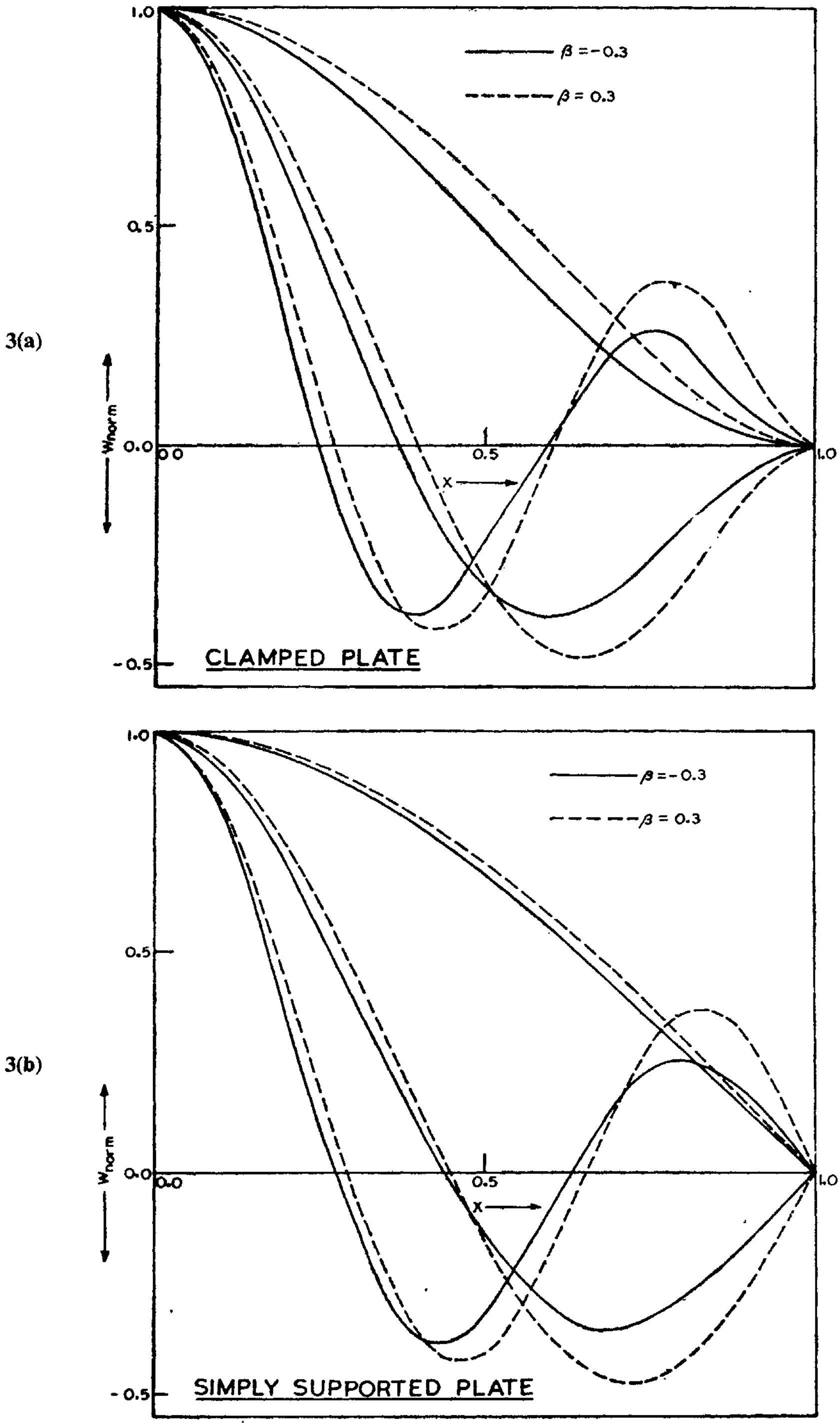

FIG. 3(a, b). Normalized transverse displacement in the first three modes of vibration for $K = 0.01$ and $\bar{N} = 2.0$.

the increased thickness of the former near the support as compared to the latter. From the plots of moments it is found that as β decreases, the lines along which the moments vanish are shifted towards the centre for both the plates.

Table I gives the critical values $\bar{N}_c$ for $\bar{N}$ for critical buckling load in compression for different foundation moduli and taper parameter β. Putting $K = 0.0$, the results agree with those of Jain (1972). It is clear from the table that the buckling

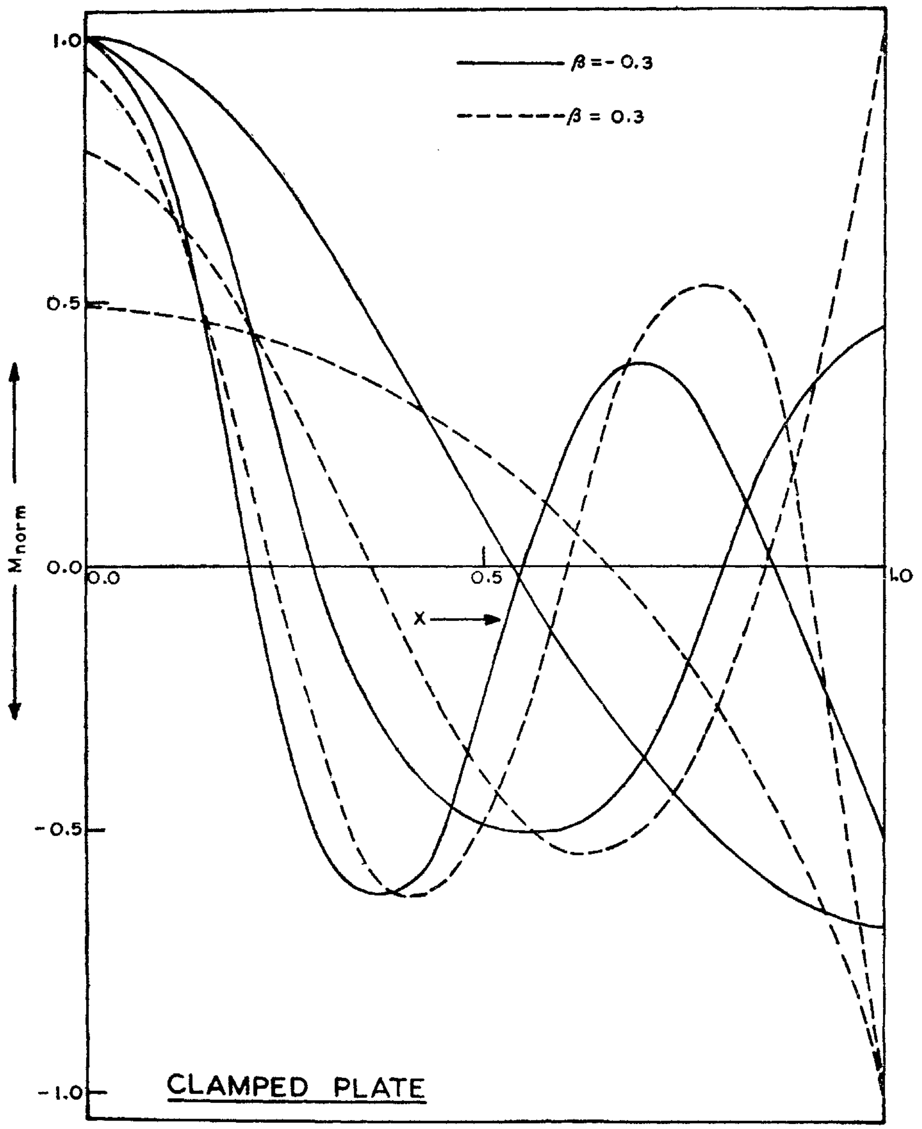

FIG. 4(a). Normalized moments in the first three modes of vibration for $K = 0.01$ and $\bar{N} = 2.0$.

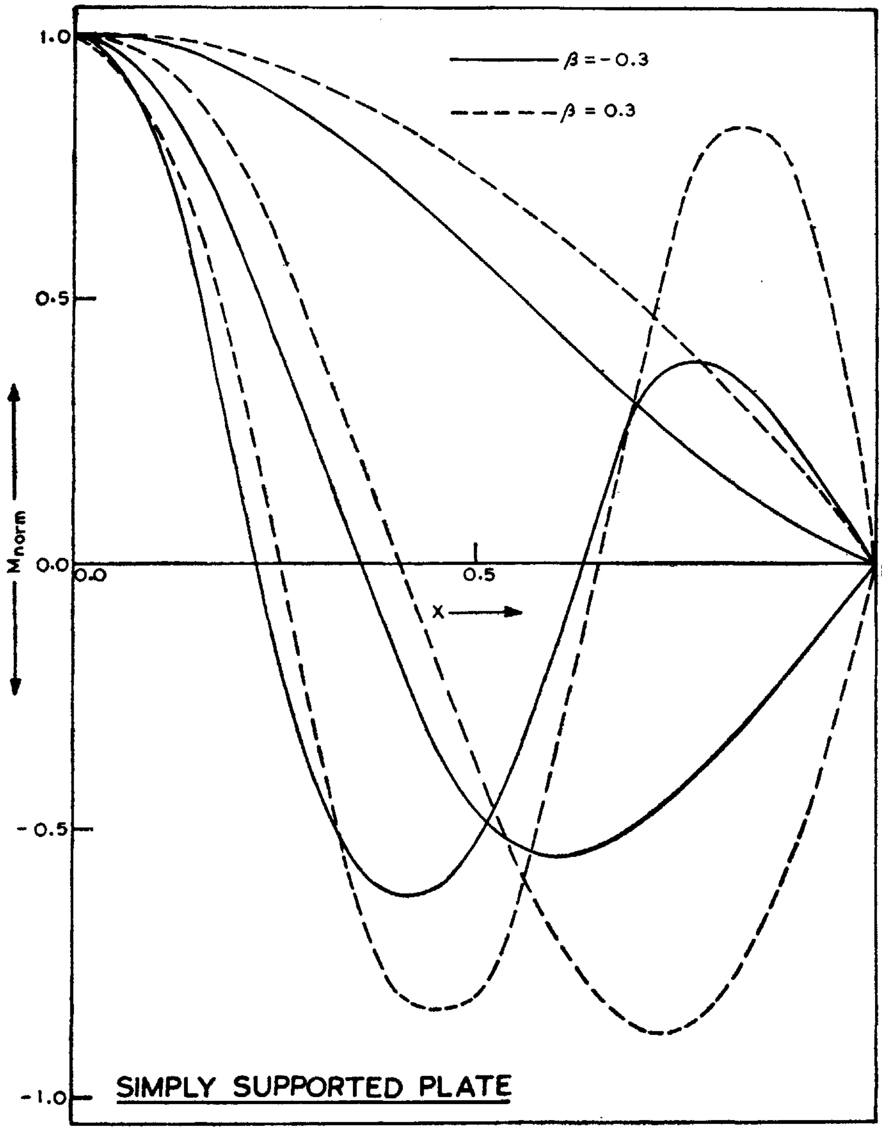

FIG. 4(b). Normalized moments in the first three modes of vibration for $K = 0.01$ and $\bar{N} = 2.0$.

load for clamped plate is larger as compared to the simply supported plate, whatever the taper constant β. The values of $\bar{N}_c$ increases continuously as β varies from positive to negative values. This may be attributed to the thickning of the plate near the edges.

TABLE I

Values of $\bar{N}$ for the critical buckling load in compression in fundamental mode $\nu = 0.3$

Values of K	Values of taper parameter β				
	− 0.3	− 0.1	0.1	0.3	0.5
Clamped					
plate 0.001	− 22.003	− 18.042	− 14.362	− 11.017	− 8.060
0.01	− 33.477	− 30.261	− 27.353	− 24.809	− 22.568
Simply supported					
plate 0.001	− 7.417	− 6.500	− 5.679	− 4.931	− 4.212
0.01	− 24.391	− 23.424	− 22.349	− 20.430	− 15.997

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