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FULLY DEVELOPED FREE CONVECTION FLOW IN A CIRCULAR PIPE

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The problem of free convection flow and heat transfer in a circular pipe using the non-linear density temperature relationship has been investigated. The results have been compared with the corresponding cases of linear and quadratic density temperature variation. It is found that for a positive γ, the characteristic parameter of the nonlinear density temperature variation, the values of velocity and temperature functions are higher than that for linear variation but smaller than that for quadratic variation whereas for a negative γ, they are lower than the corresponding ones in both the later cases. The Nusselt number has also been plotted against the free convection parameter K for various γ.

NOTATIONS

u, v, w = velocities in r, θ, z directions respectively

T = temperature

w^* = dimensionless velocity

θ = dimensionless temperature

η = dimensionless distance

β_0, β_1 = constants in (1) and (2)

γ = the NDT parameter

Nu = Nusselt number

K = dimensionless constant

K_1 = thermal conductivity

Pr = Prandtl number

α = heat source parameter

Gr = Grashof number

Subscripts

s = hydrostatic condition

w = wall condition

z = along z direction.

INTRODUCTION

Due to its wide application in the field of chemical engineering, atomic power and aeronautics, the process of natural convection flow has been studied by several authors. Ostrach (1952, 1954) first investigated the problem of laminar natural convection flow between vertical, heated walls, when the wall temperature is either constant or varying linearly along the plate length. The corresponding problem with porous walls was solved by Rao (1962). Nanda and Sharma (1963) extended the analysis to the flow in a circular pipe. However, in all these investigations, a linear density temperature variation (LDT) namely,

$$\Delta\rho = -\rho\beta(T - T_s) \qquad ...(1)$$

was taken to express the body force term as buoyancy term. Later Goren (1966) while dealing with the problem of free convection from a semi-infinite plate of uniform temperature to water at 4°C, used a quadratic density temperature distribution (QDT) viz.,

$$\Delta\rho = -\rho\beta(T - T_s)^2 \qquad ...(2)$$

where ρ is the density, β the constant and T_s the temperature in hydrostatic condition. Using relation (2), Sinha (1969) and Agarwal and Upmanyu (1976) analysed the problem of fully developed free convection flow between vertical plates and in a circular pipe respectively. Recently Sastri and Vajravelu (1977) solved the problem of free convection between vertical walls by taking the nonlinear density temperature variation (NDT) viz.,

$$\Delta\rho = -\beta_0\rho(T - T_s) - \beta_1\rho(T - T_s)^2 \qquad ...(3)$$

where β_0 and β_1 are the constants. This relation thus includes both the relationship (1) and (2). Recently Gilpin (1975) has used a density-temp. relation which is similar to relation (3) and has shown the existence of quasi-steady modes of convection for some temperature below 4°C (in case of water). A similar relation introduced by Varrier and Tien (1968) has been used to predict the heat transfer results in the case of water for temperature between 0° and 20°C. We here discuss the problem of fully developed laminar free convection flow in the presence of constant heat sources in a circular pipe, taking the same density temperature relationship (3).

It is found that the flow and heat transfer both depend upon a new parameter $\gamma = (\beta_1/\beta_0)\,\Delta T$ in addition to the heat source parameter α and the free convection parameter K.

FORMULATION OF THE PROBLEM

Consider the fully developed steady laminar free convection flow of a viscous incompressible fluid in a circular pipe. In the cylindrical coordinate system (r, φ, z), let u, v, w be the velocity components. The motion being rotationally symmetric and assuming that the pipe is long enough, all physical quantities will be independent of φ and z, and v will be zero. The corresponding equations of continuity, motion and energy are

$$\frac{\partial u}{\partial r} + \frac{u}{r} = 0 \qquad ...(4)$$

$$\rho u \frac{\partial u}{\partial r} = -\frac{\partial p}{\partial r} + \mu\left[\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{u}{r^2}\right] \qquad ...(5)$$

$$\rho u \frac{\partial w}{\partial r} = -\frac{\partial p}{\partial z} + \rho f_z + \mu\left[\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r}\right] \qquad ...(6)$$

$$\rho C_p \mu \frac{\partial T}{\partial r} = K_1\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r}\right) + \mu\left[2\left(\frac{\partial u}{\partial r}\right)^2 + 2\left(\frac{u}{r}\right)^2 + \left(\frac{\partial w}{\partial r}\right)^2\right] + Q \qquad ...(7)$$

where Q, a constant, denotes the heat added due to heat sources, f_z the generating body force, C_p the specific heat at constant pressure, K_1 the coefficient of thermal conductivity and p the pressure.

The boundary conditions are

at $\quad r = a,\ u = w = 0,\ T = T_w. \qquad ...(8)$

Following Ostrach (1952), the body force term in (6) can be expressed as buoyancy term. In the hydrostatic condition eqn. (6) gives

$$\rho_s f_z - \frac{\partial p_s}{\partial z} = 0 \qquad ...(9)$$

and hence

$$\rho f_z - \frac{\partial p}{\partial z} = (\rho - \rho_s) f_z + \rho_s f_z - \frac{\partial p}{\partial z}$$

$$= (\rho - \rho_s) f_z - \frac{\partial p_D}{\partial z} \qquad ...(10)$$

where $\quad p_D = p - p_s.$

Now using relation (3) and (10), eqns. (4) – (7) lead to

$$\frac{d^2 w}{dr^2} + \frac{1}{r}\frac{dw}{dw} + \frac{\rho}{\mu}(\beta_0\theta + \beta_1\theta^2) = 0 \qquad ...(11)$$

$$\frac{d^2\theta}{dr^2} + \frac{1}{r}\frac{d\theta}{dr} + \frac{\mu}{K_1}\left(\frac{dw}{dr}\right)^2 + \frac{\theta}{K_1} = 0, \qquad ...(12)$$

where $\quad \theta = T - T_s.$

Introducing non-dimensional variables,

$$\eta = \frac{r}{a}, \quad \theta = \frac{\theta_w \theta^*}{K_1}, \quad w = \frac{W}{K} w^*$$

where

$$\left.\begin{aligned} &\theta_w = T_w - T_s, \; W = \frac{f_z \beta_0 a^2 \theta_w}{\nu} \\ \text{and} \quad &K = \frac{f_z^2 \beta_0^2 \rho^2 a^4 \theta_w^2}{K_1}. \end{aligned}\right\} \quad \ldots(13)$$

Equations (11) and (12) reduce to

$$\frac{d^2 w^*}{d\eta^2} + \frac{1}{\eta}\frac{dw^*}{d\eta} + \theta^* + \frac{\theta^{*2}}{K}\gamma = 0 \quad \ldots(14)$$

$$\frac{d^2\theta^*}{d\eta^2} + \frac{1}{\eta}\frac{d\theta^*}{d\eta} + \left(\frac{dw^*}{d\eta}\right)^2 + \alpha K = 0 \quad \ldots(15)$$

where $\alpha (= Qa^2/\theta_w K_1)$ is the heat source parameter. The parameter K can also be expressed as $K = Gr(Pr) f_z a/C_p$, in which $Gr (= \beta_0 f_z a^3 \theta_w^2/\nu^2)$ is the Grash of number and Pr is the Prandtl number.

For the sake of convenience, dropping the stars, eqns. (14) and (15) finally are

$$\frac{d^2 w}{d\eta^2} + \frac{1}{\eta}\frac{dw}{d\eta} + \theta + \frac{\gamma}{K}\theta^2 = 0 \quad \ldots(16)$$

$$\frac{d^2\theta}{d\eta^2} + \frac{1}{\eta}\frac{d\theta}{d\eta} + \left(\frac{dw}{d\eta}\right)^2 + \alpha K = 0. \quad \ldots(17)$$

The corresponding boundary conditions are

$$\left.\begin{aligned} &\text{at } \eta = 0, \frac{dw}{d\eta} = 0, \; \frac{d\theta}{d\eta} = 0 \\ &\text{at } \eta = 1, \; w = 0, \; \theta = K. \end{aligned}\right\} \quad \ldots(18)$$

SOLUTION

For the solution of eqns. (16) and (17), we assume,

$$w = Kw_0 + K^2 w_1 + K^3 w_2 + \ldots \quad \ldots(19)$$

$$\theta = K\theta_0 + K^2\theta_1 + K^3\theta_2 + \ldots. \quad \ldots(20)$$

Substituting (19) and (20) into (16) and (17) and equating the coefficients of like powers of K on either side of the equations thus obtained, we get the following set of equations

$$\left.\begin{aligned} w_0'' + \frac{1}{\eta} w_0' + \theta_0 + \gamma \theta_0^2 &= 0 \\ w_1'' + \frac{1}{\eta} w_1' + \theta_1 + 2\gamma \theta_0 \theta_1 &= 0 \\ w_2'' + \frac{1}{\eta} w_2' + \theta_2 + \gamma(2\theta_0\theta_2 + \theta_1^2) &= 0 \end{aligned}\right\} \quad \ldots(21)$$

$$\left.\begin{aligned} \theta_0'' + \frac{1}{\eta} \theta_0' + \alpha &= 0 \\ \theta_1'' + \frac{1}{\eta} \theta_1' + w_0'^2 &= 0 \\ \theta_2'' + \frac{1}{\eta} \theta_2' + 2\gamma w_0' w_1' &= 0. \end{aligned}\right\} \quad \ldots(22)$$

The boundary conditions (18) then reduce to

$$\text{at} \quad \left.\begin{aligned} \eta = 0,\ w_0' = w_1' = w_2' &= 0 \\ \theta_0' = \theta_1' = \theta_2' &= 0 \end{aligned}\right\} \quad \ldots(23)$$

$$\text{at} \quad \left.\begin{aligned} \eta = 1,\ w_0 = w_1 = w_2 &= 0 \\ \theta_0 = 1,\ \theta_2 = \theta_3 &= 0. \end{aligned}\right\} \quad \ldots(24)$$

Solving eqns. (21) and (22) under (23) and (24), we get

$$w_0 = D_0 - A_1\eta^2 + A_2\eta^4 - A_3\eta^6 \quad \ldots(25)$$

$$\theta_0 = 1 + \alpha/4 - \alpha\eta^2/4 \quad \ldots(26)$$

$$w_1 = E_1 - E_3\eta^{16} + E_4\eta^{14} - E_5\eta^{12} + E_6\eta^{10} - E_7\eta^8 + E_9\eta^4 - E_{10}\eta^2 \quad \ldots(27)$$

$$\theta_1 = D_1 - A_4\eta^{12} + A_5\eta^{10} - A_6\eta^8 + A_7\eta^6 - A_8\eta^4 \quad \ldots(28)$$

where D_0, D_1, E_1, $A_1 \ldots A_8$, $E_3 \ldots E_{10}$ all are constants and depend upon γ and α. The expressions for θ_2 and w_2 have also been obtained but, being long enough, are not given here. The velocity and temperature functions are then obtained from (19) and (20). The rate of heat transfer from the pipe wall to the fluid per unit area of the pipe surface is given by

$$q = \frac{K_1\theta_w}{aK}\left(\frac{\partial\theta}{\partial\eta}\right)_{\eta=1}. \quad \ldots(29)$$

The Nusselt number is therefore

$$Nu = \frac{qa}{\theta_w K_1} = \frac{1}{K}\left(\frac{\partial\theta}{\partial\eta}\right)_{\eta=1}. \quad \ldots(30)$$

NUMERICAL RESULTS

The numerical values of velocity and temperature functions have been computed for $K = 0.5$, $\alpha = 5$, 10 and $\gamma = -0.2$, 0.05 and 0.2. For the sake of comparison, the corresponding values for same K and α for relationship (1) and (2) have been calculated with the results of Nanda and Sharma (1963) and Agarwal and Upmanyu (1976), respectively. All the numerical results thus obtained are shown in Figs. 1 – 4. It is evident that both velocity and temperature increase uniformly from the wall to the centre of the pipe for all γ. It is further observed that for $\alpha = 5$, the velocity and temperature near the centre of the pipe are considerably large for positive γ, the NDT parameter and small for negative γ as compared to LDT, while both are less than QDT irrespective of the sign of γ. The situation is more pronounced for $\alpha = 10$.

The Nusselt number Nu is plotted against K in Figs. 5 and 6. It is found that it remains negative and numerically increases with K irrespective of the values of α and γ, thus showing that the wall gets more and more heated with increase in the parameters. It is interesting to note that for a positive γ, both for $\alpha = 5$ and $\alpha = 10$, the rate of heating to the wall is much enhanced at high values of K.

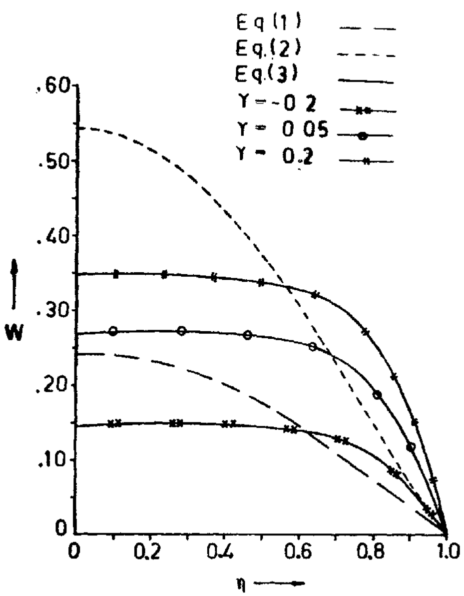

FIG. 1. Dimensionless velocity versus η for $\alpha = 5$ and $K = 0.5$.

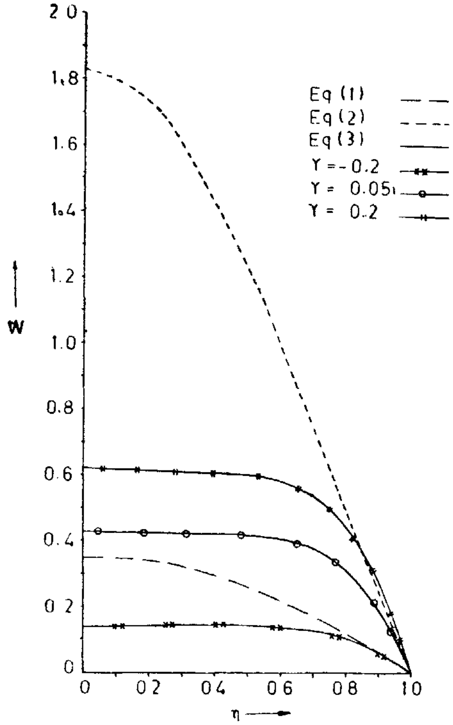

FIG. 2. Dimensionless velocity versus η for $\alpha = 10$ and $K = 0.5$.

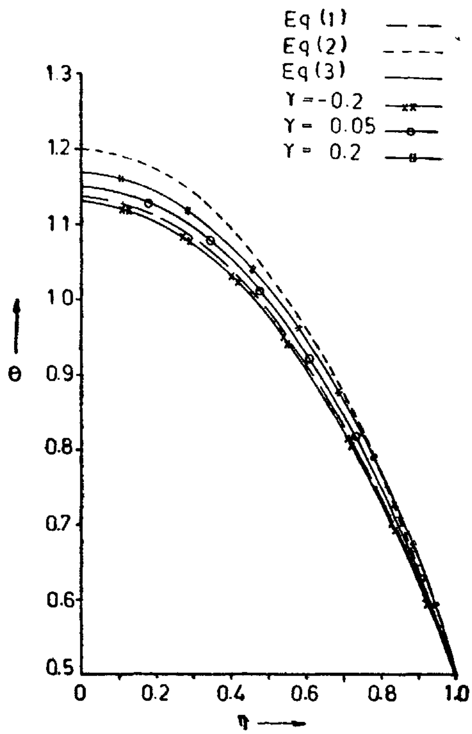

FIG. 3. Dimensionless temperature versus η for $\alpha = 5$ and $K = 0.5$.

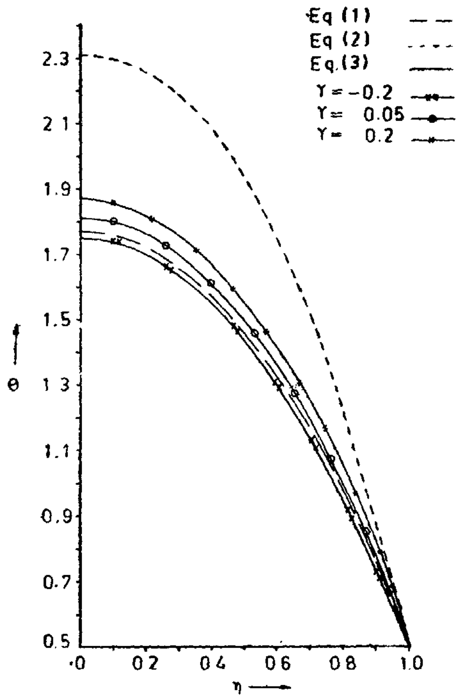

FIG. 4. Dimensionless temperature versus η for $\alpha = 10$ and $K = 0.5$.

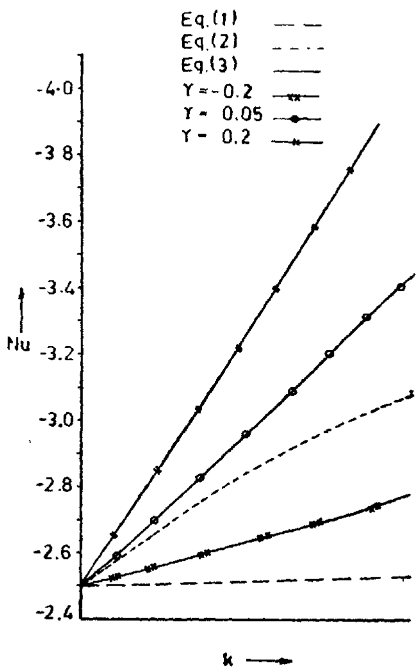

FIG. 5. Nusselt number versus K for $\alpha = 5$.

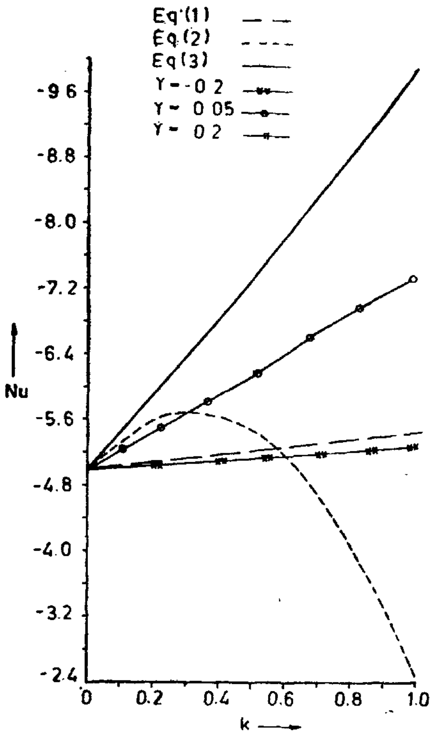

FIG. 6. Nusselt number versus K for $\alpha = 10$.

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