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INSTABILITY ZONES DUE TO ACOUSTIC PULSE STRIKING A VORTEX SHEET ON A SUPERSONIC STREAM

H. GHOSH

Department of Mathematics, Bengal Engineering College, Howrah 711103, *West Bengal*

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The propagation of sound across a vortex sheet separating two fluids in relative supersonic motion, but with the same density and sound speed, is considered for an acoustic line source in the form of a pulse in the moving medium. It is found that instability waves depending on M, the Mach number, appear in specified zones of the ambient medium.

1. INTRODUCTION

Jones and Morgan (1972) and Hardisty (1972/73) discussed the problem of the effect of a sound source on a vortex sheet that separates two fluids in relative subsonic motion. While Jones and Morgan placed the source in the stationary medium, Hardisty took it in the moving fluid. It was shown by them that after a length of time a singularity arises in the reflected wave which initiates the instability wave in the initial value problem if the acoustic source is taken as a two dimensional point source of a pulse type. They also discussed harmonic solution which contained terms that grow exponentially with distance downstream and that tend to concentrate energy downstream.

In another paper, Jones and Morgan (1972/73) considered the problem of instability due to acoustic radiation striking a vortex sheet on a supersonic stream when the source was placed in the stationary medium. The purpose of this paper was to examine how the results obtained by them in the subsonic case, change when the stream is taken to be supersonic. It was shown by them that instability waves are again initiated if the Mach number M is less than $2\sqrt{2}$ and that they are absent if M is greater than $2\sqrt{2}$. It was shown further that some new waves appear if $M > 2$. One comes into existence if $M > 2$ and another two arise if $M > 2\sqrt{2}$. They used the method of Fourier transforms to tackle the harmonic problem and obtained solutions for a wide range of Mach numbers. But they did not emphasise all the effects of direction or the observation position with reference to the direction of the motion of stream and this feature obviously should have experienced instability in certain cases. The author of this paper has emphasised on this phenomenon when the acoustic wave is incident in a particular direction from the moving medium. The case in which the acoustic source is placed in this medium has probably more physical significance than the one in which it is placed in the ambient fluid, since a jet may be expected to generate sources within itself. The source is taken to be time dependent other than a harmonic one. Emphasis has been laid on the instability to be experienced in the ambient medium.

The method of Laplace transform is used to tackle the problem which is formulated in section 2. The solution has been obtained in the form of a complex integral similar to that of Fourier transform. In section 3 the contour of integration has been deformed [cf. Noble 1958] and the integration along the deformed contour is carried with respect to a new variable obtained by suitable substitution. This has been done in order to study closely the field in the still medium.

2. Formulation of the Problem

The problem to be considered is that of a fluid moving supersonically and separated from a still medium by a plane vortex sheet. The plane of the vortex sheet is taken to be the plane $y = 0$, the ambient fluid occupying $y < 0$ and the moving stream $y > 0$. The velocity of the stream is of magnitude U and is parallel to the x-axis. The motion is taken to be in two dimensions in the xy-plane. Let the moving medium contain the acoustic source in the form of pulse.

The effects of viscosity, thermal conductivity and gravity are neglected and both fluids are assumed to have the same constant density and the same sound speed a. The analysis is substantially more complicated when the densities and the sound speeds of the two media are different, but the basic phenomena are not significantly affected.

After linearization, the equations of motion governing the flow are

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} - \left(M\frac{\partial}{\partial x} + \frac{1}{a}\frac{\partial}{\partial t}\right)^2 \phi = 0 \quad (y > 0) \qquad \ldots(1)$$

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} - \frac{1}{a^2}\frac{\partial^2\phi}{\partial t^2} = 0 \quad (y < 0) \qquad \ldots(2)$$

where $M = \dfrac{U}{a}\cdot$

The continuity of pressure requires that on $y = 0$

$$\left(M\frac{\partial}{\partial x} + \frac{1}{a}\frac{\partial}{\partial t}\right)\phi(x, +0, t) = \frac{1}{a}\frac{\partial}{\partial t}\phi(x, -0, t) \qquad \ldots(3)$$

while the continuity of particle displacement necessitates

$$\frac{1}{a}\frac{\partial}{\partial t}\frac{\partial}{\partial y}\phi(x, +0, t) = \left(M\frac{\partial}{\partial x} + \frac{1}{a}\frac{\partial}{\partial t}\right)\frac{\partial}{\partial y}\phi(x, -0, t). \qquad \ldots(4)$$

The incident field in $y > 0$ is taken to be

$$\phi_{inc} = \begin{cases} \dfrac{\exp\{at - \mu x\cos\alpha + y\sin\alpha\}}{\{at - \mu x\cos\alpha + y\sin\alpha\}^{1/2}}, & \text{for } t > \dfrac{\mu x\cos\alpha - y\sin\alpha}{a} \\ 0, & \text{for } t \leqslant \dfrac{\mu x\cos\alpha - y\sin\alpha}{a} \end{cases} \quad \ldots(5)$$

so that ϕ and $\dfrac{\partial\phi}{\partial t}$ can be taken to vanish at time $t = 0$. The condition

$$\mu\cos\alpha = \frac{M + \sqrt{1 + (M^2 - 1)\sin^2\alpha}}{M^2 - 1} \quad \ldots(6)$$

is imposed since it requires (5) to satisfy (1).

Write

$$\bar{\phi}(x, y, p) = \int_0^\infty \phi(x, y, t)\, e^{-pt}\, dt.$$

Then (1) and (2) give

$$\left[(M^2 - 1)\frac{\partial^2\bar{\phi}}{\partial x^2} + 2M \,.\, \frac{p}{a}\frac{\partial\bar{\phi}}{\partial x} + \frac{p^2}{a^2}\bar{\phi}\right] - \frac{\partial^2\bar{\phi}}{\partial y^2} = 0 \ (y > 0) \quad \ldots(7)$$

$$\left[\frac{\partial^2\bar{\phi}}{\partial x^2} - \frac{p^2}{a^2}\bar{\phi}\right] + \frac{\partial^2\bar{\phi}}{\partial y^2} = 0 \ (y < 0). \quad \ldots(8)$$

Also (3) and (4) give

$$M\frac{\partial}{\partial x}\bar{\phi}(x, +0, p) + \frac{p}{a}\bar{\phi}(x, +0, p) = \frac{p}{a}\bar{\phi}(x, -0, p) \quad \ldots(9)$$

$$\frac{p}{a}\frac{\partial}{\partial y}\bar{\phi}(x, +0, p) = M\frac{\partial^2}{\partial x\,\partial y}\bar{\phi}(x, -0, p) + \frac{p}{a}\frac{\partial}{\partial y}\bar{\phi}(x, -0, p). \quad \ldots(10)$$

Further (5) gives

$$\bar{\phi}_{inc} = \frac{1}{a}\sqrt{\frac{\pi}{(p/a) - 1}} \,.\, H(x) \,.\, \exp\left(-\frac{p}{a}(\mu x\cos\alpha - y\sin\alpha)\right) \quad \ldots(11)$$

where

$$H(x) = \begin{cases} 1, & \text{for } x > 0 \\ 0, & \text{for } x < 0. \end{cases}$$

The general solutions of (7) and (8) are easily written down and, after inserting the incident field, one readily gets

$$\bar{\phi} = \frac{1}{a}\sqrt{\frac{\pi}{\frac{p}{a}-1}} \,.\, H(x) \,.\, \exp\left(- \frac{p}{a}(\mu x \cos\alpha - y \sin\alpha)\right)$$

$$+ \int_{C_1} A(\xi) \,.\, \exp\left(- \frac{p}{a}[\{1 - (M^2 - 1)\,\xi^2$$

$$+ 2i\,M\xi\}^{1/2} \,.\, y - i\xi \,.\, x]\right) d\xi \;(y > 0) \qquad \ldots(12)$$

$$\bar{\phi} = \int_{C_2} B(\xi) \,.\, \exp\left(\frac{p}{a}\,[\{1 + \xi^2\}^{1/2} \,.\, y + i\xi \,.\, x]\right) d\xi \;(y < 0) \qquad \ldots(13)$$

where the contours C_1 and C_2 are at one's disposal. The vanishing conditions at infinity are satisfied in both (12) and (13) for all real ξ.

$A(\xi)$ and $B(\xi)$ may be obtained by substituting (12) and (13) in (9) and (10), identifying C_1 and C_2 with a common contour which is taken to be the straight line from $\xi = -\infty$ to $\xi = +\infty$ and taking Fourier inverse transforms of the resulting equations. One thus finally gets

$$\bar{\phi} = \frac{1}{a}\sqrt{\frac{\pi}{\frac{p}{a}-1}} \,.\, H(x) \,.\, \exp\left(- \frac{p}{a}(\mu x \cos\alpha - y \sin\alpha)\right)$$

$$+ \int_{-\infty}^{\infty} A(\xi) \,.\, \exp\left(- \frac{p}{a}\,[wy - i\xi\, x]\right) d\xi \;(y > 0) \qquad \ldots(14)$$

$$\bar{\phi} = \int_{-\infty}^{\infty} B(\xi) \,.\, \exp\left(\frac{p}{a}\,[vy + i\xi x]\right) d\xi \;(y < 0) \qquad \ldots(15)$$

with

$$A(\xi) = \frac{1}{2a}\sqrt{\frac{1}{\pi\left(\frac{p}{a}-1\right)}}$$

$$\times \frac{\sin\alpha + (M\mu\cos\alpha - 1)\,\{1 + \xi^2\}^{1/2}\,(1 + iM\,\xi)}{(\mu\cos\alpha + i\xi)\,.\,\Lambda(\xi)} \qquad \ldots(16)$$

$$B(\xi) = \frac{1}{2a}\sqrt{\frac{1}{\pi\left(\frac{p}{a}-1\right)}}$$

$$\times \frac{(1 + iM\,\xi)\sin\alpha - (M\mu\cos\alpha - 1)\,\{1 - (M^2 - 1)\,\xi^2 + 2iM\,\xi\}^{1/2}}{(\mu\cos\alpha + i\xi)\,.\,\Lambda(\xi)} \qquad \ldots(17)$$

where

$$\Lambda(\xi) = \{1 + \xi^2\}^{1/2} (1 + iM\xi)^2 + \{1 - (M^2 - 1)\xi^2 + 2iM\xi\}^{1/2} \qquad ...(18)$$

and

$$v^2 = 1 + \xi^2, \qquad w^2 = (1 + iM\xi)^2 + \xi^2.$$

Let v and w be defined by choosing those branches which are unity at $\xi = 0$. The branch lines for v are along the imaginary axis from $-i$ to $-i\infty$ and from $+i$ to $+i\infty$ respectively. Since attention is restricted to the supersonic case $M > 1$, the branch lines for w are along the imaginary axis joining the points $\xi = i/(M+1)$ and $\xi = i/(M-1)$. The branch lines are displayed in Fig. 1 for the two cases, $1 < M < 2$ and $2 < M$.

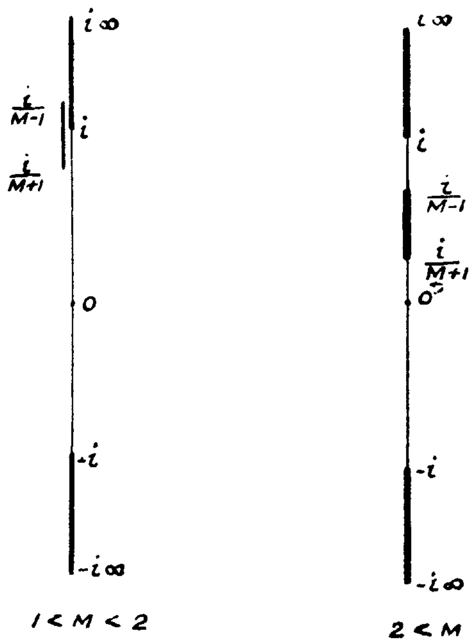

FIG. 1.

Following Jones and Morgan (1972/73) the zeros of $\Lambda(\xi)$ are found to be as follows :

When $1 < M < 2$, $\Lambda(\xi)$ has two complex zeros

$$\xi = \frac{i}{2}(q_1 \pm iq_2)$$

where

$$q_1 = \frac{1}{M} + \sqrt{\left(1 + \frac{1}{M^2}\right)}, \qquad q_2 = \left(\frac{2q_1}{M} - 1\right)^{1/2}.$$

When $2 < M < 2\sqrt{2}$, $\Lambda(\xi)$ has the same two complex zeros and also the imaginary zero $\xi = 2i/M$.

When $2\sqrt{2} < M$, $\Lambda(\xi)$ has three imaginary zeros $\xi = 2i/M$, $\xi = \frac{1}{2}i\,(q_1 \pm iq_2)$ which all lie between $\xi = i/(M-1)$ and $\xi = i$, the middle one of the three being $\xi = 2i/M$.

3. Solution in the Ambient Medium

To study the field given by (15) introduce the notation $x = r\cos\theta$, $y = -r\sin\theta$, $0 < \theta < \pi$. In $0 < \theta < \pi$ take

$$\xi = i\cos(\theta + i\lambda),\ -\infty < \lambda < \infty \qquad \text{...(19)}$$

so that

$$\nu y + i\xi x = -r\cosh\lambda.$$

Now θ = constant defines a hyperbola. As λ goes from $-\infty$ to $+\infty$, the upper or the lower branch of the hyperbola $\xi = i\cos(\theta + i\lambda)$ is described in the sense of the arrow shown in Fig. 2 according as $0 < \theta < \pi/2$ or $\pi/2 < \theta < \pi$.

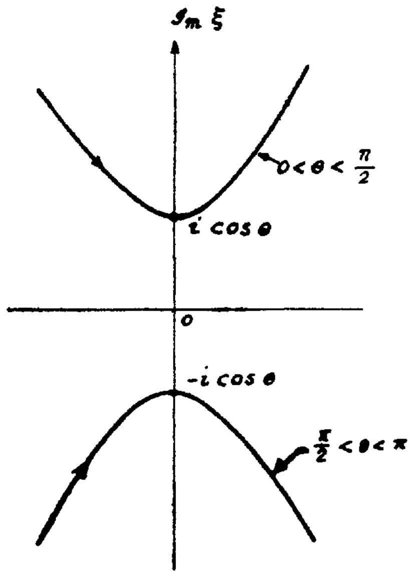

Fig. 2.

Now deform the contour as shown in the figures below on to the appropriate half of the hyperbola and when necessary, with proper deviations so as not to encounter any branch cut in the ξ-plane. In the usual way connect the two contours by arcs at infinity and, in view of the vanishing behaviour of the integrand at infinity, the contributions from these arcs are zero. In the deformation the contributions of the poles that are to be crossed must be given due considerations.

When the contour is shifted by (19), the positions of the poles with respect to the closed contour are studied now.

Case I : The region $x > 0$

This region is defined by $0 < \theta < \frac{\pi}{2}$. Subdivide this region into two parts:

$$0 < \theta < \cos^{-1}\frac{1}{M+1} \text{ and } \cos^{-1}\frac{1}{M+1} < \theta < \frac{\pi}{2}.$$

Later, one shall have occasion to further subdivide the first subinterval into two :

$$0 < \theta < \cos^{-1}\frac{1}{M-1} \text{ and } \cos^{-1}\frac{1}{M-1} < \theta < \cos^{-1}\frac{1}{M+1}.$$

The inequalities, for all $M > 1$, may first be noted :

$$\frac{1}{M+1} < \frac{1}{\sqrt{2}}, \quad \frac{1}{M+1} < \frac{1}{M-1} \leqslant \mu\cos\alpha \leqslant \frac{2M}{M^2-1}$$

and

$$\frac{2M}{M^2-1} > 1 \quad \text{for} \quad 1 < M < \sqrt{2}+1$$

$$< 1 \quad \text{for} \quad \sqrt{2}+1 < M.$$

If $\mu\cos\alpha > 1$, the pole $\xi = i\mu\cos\alpha$ falls outside the closed contour. This pole falls inside if and only if $\mu\cos\alpha < \cos\theta < 1$.

To study the location of the complex poles $\xi = \frac{i}{2}(q_1 \pm iq_2)$ for $M < 2\sqrt{2}$ with respect to the contour, it is only necessary to find the position with respect to the hyperbola

$$\frac{\tau^2}{\cos^2\theta} - \frac{\sigma^2}{\sin^2\theta} = 1$$

where $\xi = \sigma + i\tau$. It is found that the poles are within the contour if $0 < \theta < \pi/4$ and outside it if $\pi/4 < \theta < \pi/2$. One then gets the following result regarding the position of poles.

(A) *When* $0 < \theta < \cos^{-1}\frac{1}{M+1}$

(a) When $0 < \theta < \cos^{-1}\frac{1}{M+1}$ and $1 < M < 2$, the imaginary pole $\xi = i\mu\cos\alpha$ is outside the closed contour. The complex poles $\xi = \frac{1}{2}i(q_1 \pm iq_2)$

are inside if $0 < \theta < \pi/4$ and outside if $\pi/4 < \theta < \cos^{-1}\dfrac{1}{M+1}$. The imaginary pole $\xi = 2i/M$ falls outside since $2/M > 1$. The deformed contour is as shown in Fig. 3.

(b-i) When $0 < \theta < \cos^{-1}\dfrac{1}{M-1}$ and $2 < M < 2\sqrt{2}$, the imaginary pole $\xi = i\mu\cos\alpha$ falls inside if $0 < \theta < \cos^{-1}\dfrac{M + \sqrt{1 + (M^2 - 1)\sin^2\alpha}}{M^2 - 1}$ and outside otherwise. The complex poles $\xi = \frac{1}{2}i(q_1 \pm iq_2)$ fall within the contour if $2 < M < \sqrt{2} + 1$. They also fall within if $\sqrt{2} + 1 < M < 2\sqrt{2}$ together with $0 < \theta < \pi/4$. They are outside otherwise. The imaginary pole $\xi = 2i/M$ falls inside if $0 < \theta < \cos^{-1}\dfrac{2}{M}$ and outside otherwise. The deformed contour is as shown in Fig. 4.

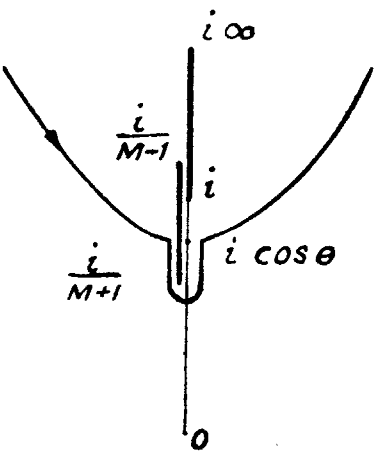

FIG. 3.

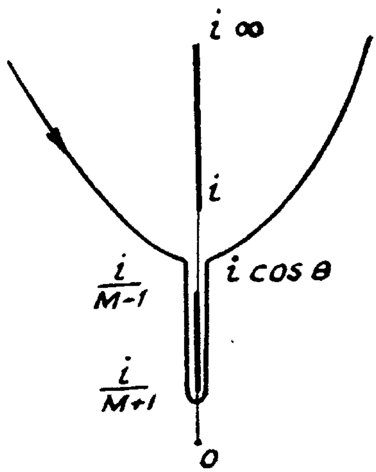

FIG. 4.

(b-ii) When $\cos^{-1}\dfrac{1}{M-1} < \theta < \cos^{-1}\dfrac{1}{M+1}$ and $2 < M < 2\sqrt{2}$, the imaginary pole $\xi = i\mu\cos\alpha$ is outside the closed contour. The complex poles $\xi = \frac{1}{2}i(q_1 \pm iq_2)$ fall inside the contour if $2 < M < \sqrt{2} + 1$ together with $\cos^{-1}\dfrac{1}{M-1} < \theta < \dfrac{\pi}{4}$ and outside otherwise (i.e. when $\dfrac{\pi}{4} < \theta < \cos^{-1}\dfrac{1}{M+1}$ or when $\sqrt{2} + 1 < M < 2\sqrt{2}$). The imaginary pole $\xi = 2i/M$ remains outside the contour. The deformed contour is as shown in Fig. 5.

(c) If $0 < \theta < \cos^{-1}\dfrac{1}{M+1}$ and $2\sqrt{2} < M$, all the four poles are imaginary. The imaginary pole $\xi = i\mu\cos\alpha$ falls inside the contour if $0 < \theta < \cos^{-1}\dfrac{M+\sqrt{1+(M^2-1)\sin^2\alpha}}{M^2-1}$ and outside otherwise. The remaining three poles are $\xi = \dfrac{i}{2}\left(q_1 + \sqrt{1-\dfrac{2q_1}{M}}\right), \xi = \dfrac{2i}{M}, \xi = \dfrac{i}{2}\left(q_1 - \sqrt{1-\dfrac{2q_1}{M}}\right)$ and they are all above the point $\xi = 0$. All of them fall within the contour if and only if $0 < \theta < \cos^{-1}\frac{1}{2}\left(q_1 + \sqrt{1-\dfrac{2q_1}{M}}\right)$. Only the last two fall within if and only if $\cos^{-1}\frac{1}{2}\left(q_1 + \sqrt{1-\dfrac{2q_1}{M}}\right) < \theta < \cos^{-1}\dfrac{2}{M}$. Only the last one falls within if and only if $\cos^{-1}\dfrac{2}{M} < \theta < \cos^{-1}\frac{1}{2}\left(q_1 - \sqrt{1-\dfrac{2q_1}{M}}\right)$.

(B) *When* $\cos^{-1}\dfrac{1}{M+1} < \theta < \dfrac{\pi}{2}$.

(d) When $\cos^{-1}\dfrac{1}{M+1} < \theta < \dfrac{\pi}{2}$ and $1 < M$, no pole falls inside the contour. Also no branch cut is encountered. The contour is as shown in Fig. 6.

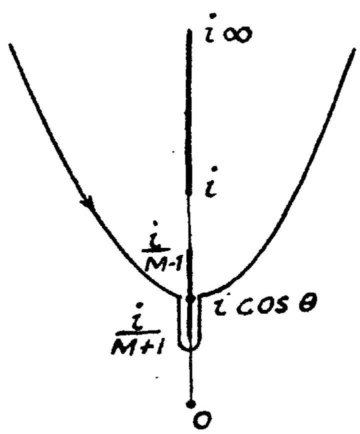

FIG. 5.

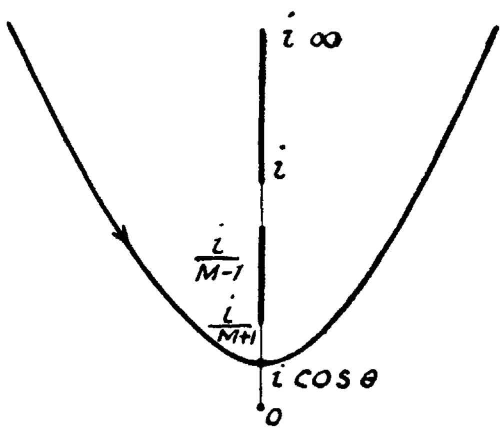

FIG. 6.

Case II : *The region* $x < 0$

This region is defind by $\dfrac{\pi}{2} < \theta < \pi$.

All the poles fall outside the closed contour. The deformed contour is as shown in Fig. 7.

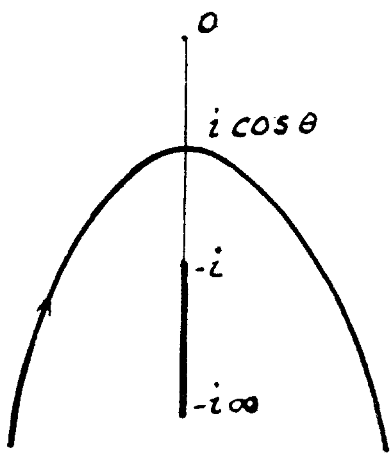

FIG. 7.

Returning now to eqn. (15), it is seen that

$$\bar{\phi} = \bar{\phi}_1 + \bar{\phi}_2 + \bar{\phi}_3 \qquad \text{...(20)}$$

where

$$\bar{\phi}_1 = \int_{-\infty}^{\infty} B(i\cos(\theta + i\lambda)) \cdot \sin(\theta + i\lambda) \cdot \exp\left(- \frac{p}{a} \cdot r\cosh\lambda \right) d\lambda,$$

$\bar{\phi}_2$ = sum of terms arising out of the contributions from poles, if any, other than the imaginary ones

$$\bar{\phi}_3 = \frac{i}{a}\sqrt{\frac{\pi}{(p-a)/a}} \times \sum \frac{(1 - M\gamma)\sin\alpha - (M\mu\cos\alpha - 1)\sqrt{(1 - M\gamma)^2 - \gamma^2}}{F(i\gamma)} \times \exp\left(\frac{p}{a}\left[\sqrt{1-\gamma^2} \cdot y - \gamma x\right]\right)$$

$$F(i\gamma) = \lim_{\xi \to i\gamma} \frac{(\mu\cos\alpha + i\xi)\,\Lambda(\xi)}{\xi - i\gamma}$$

$i\gamma$ being an imaginary pole, if any, satisfying $0 < \gamma < \cos\theta$ and the summation running over all possible values of γ.

Now that only $\bar{\phi}_3$ is the contribution of the imaginary poles, the other terms in (20) do not warrant discussion. Instability waves will make themselves manifest in specified parts of the ambient medium and, for this, one has

$$\phi_3 = i \sum \frac{(1 - M\gamma)\sin\alpha - (M\mu\cos\alpha - 1)\sqrt{(1 - M\gamma)^2 - \gamma^2}}{F(i\gamma)}$$

$$\times \frac{H(at + \sqrt{1 - \gamma^2}\cdot y - \gamma x)}{\sqrt{(at + \sqrt{1 - \gamma^2}\cdot y - \gamma x)}} \times \exp(at + \sqrt{1 - \gamma^2}\cdot y - \gamma x).$$

The regions of the ambient medium in which instability occurs for different values of M are those wedge-shaped regions described in section 3 Case I A (b-i), (b-ii), (c) for which imaginary poles fall within the contour. For example, if the supersonic flow is defined by $M = 4/\sqrt{3}$, the region of instability will be the wedge $0 < \theta < \pi/6$ for all directions of incidence α.

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