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APPROXIMATION OF A FUNCTION BY A SINGULAR INTEGRAL IN L_1-NORM

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In the present paper we generalize a theorem established by Butzer (1960) on approximation of a function using the concepts of a general singular integral.

§1. The general singular integral associated with a Lebesgue integrable function $f(x)$ over $(-\infty, \infty)$ is given by

$$J_e(x) = \frac{e}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x+u)\, K(eu)\, du$$

where e is a continuous parameter and $K(u)$ is a kernel, which satisfied the following conditions:

(a) $K(u)$ is a non-negative function of the real variable u, $-\infty < u < \infty$, such that $K(u) \in L_1(-\infty, \infty)$.

(b) $K(u)$ is continuous and even function.

(c) $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} K(u)\, du = 1$

or (c*) There exists a majorant k^* of K such that it satisfies (c).

(d) $\mu_\beta = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} u^\beta K(u)\, du < \infty, \ \beta > 0.$

In a recent paper Butzer (1960) has shown that if $f \in L(-\infty, \infty)$, $K(u)$ satisfies conditions (a) – (d) and

$$\omega(u) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |\, g(x; u) \,|\, dx \qquad ...(1.1)$$

where

$$g(x; u) = f(x+u) + f(x-u) - 2f(x) \qquad ...(1.2)$$

then the condition

$$\int_0^t \omega(u)\, du = O(t^{\gamma+1}),\ t \to 0 \qquad \text{...(1.3)}$$

implies that

$$\| J_e(x) - f(x) \|_{L_1} = O(e^{-\gamma}),\ e \uparrow \infty,\ \gamma > 0 \qquad \text{...(1.4)}$$

where the $\| \ \|_{L_1}$ is defined for all $f(x) \in L_1(-\infty, \infty)$ by

$$\| f(x) \|_{L_1} = \int_{-\infty}^{\infty} | f(x) |\, dx < \infty.$$

The object of this paper is to generalize the theorem of Butzer (1960). We in fact, prove the following:

Theorem — If $f(x) \in L_1(-\infty, \infty)$, $K(u)$ satisfies the conditions (a) - (d) and

$$\int_0^t \omega(u)\, du = O[th(t)] \qquad \text{...(1.5)}$$

then

$$\| J_e(x) - f(x) \|_{L_1} = O[h(1/e)]$$

where $h(t)$ is a positive function of t such that

$$h(t) \to 0,\ \text{as}\ t \to 0 \qquad \text{...(1.6)}$$

$h(t)$ is monotonic increasing in $(0, \delta)$, δ being small, but fixed.

It may be mentioned here that the theorem of Butzer (1960) is a particular case of our theorem for $h(t) = t^{\gamma}$.

§2. *Proof of the theorem* — We have

$$J_e(x) = \frac{e}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x+u)\, K(eu)\, du$$

$$= \frac{e}{\sqrt{2\pi}} \int_0^{\infty} \{f(x+u) + f(x-u)\}\, K(eu)\, du.$$

Putting $f(x) = 1$, for all $x \in (-\infty, \infty)$, we obtain

$$J_e(x) = \frac{e}{\sqrt{2\pi}} \int_{-\infty}^{\infty} K(eu)\, du = 1.$$

Hence, we have

$$f(x) = \frac{e}{\sqrt{2}\,\pi} \int_{-\infty}^{\infty} f(x)\, K(eu)\, du$$

$$= \frac{2e}{\sqrt{2}\,\pi} \int_{0}^{\infty} f(x)\, K(eu)\, du.$$

Therefore

$$J_e(x) - f(x) = \frac{e}{\sqrt{2}\,\pi} \int_{0}^{\infty} \{f(x+u) + f(x-u) - 2f(x)\}\, K(eu)\, du$$

$$= \frac{e}{\sqrt{2}\,\pi} \int_{0}^{\infty} g(x;u)\, K(eu)\, du.$$

Now, we have

$$\| J_e(x) - f(x) \|_{L_1} = \int_{-\infty}^{\infty} | J_e(x) - f(x) |\, dx$$

$$= \int_{-\infty}^{\infty} \left| \frac{e}{\sqrt{2}\pi} \int_{0}^{\infty} g(x;u)\, K(eu)\, du \right| dx.$$

$$\leqslant e \int_{0}^{\infty} | K(eu) |\, du \frac{1}{\sqrt{2}\,\pi} \int_{-\infty}^{\infty} | g(x,u) |\, dx$$

$$= e \int_{0}^{\infty} \omega(u)\, K(eu)\, du$$

$$= e \left\{ \int_{0}^{1/e} + \int_{1/e}^{\delta} + \int_{\delta}^{\infty} \right\} \omega(u)\, K(eu)\, du$$

$$= I_1 + I_2 + I_3, \text{ say} \qquad \ldots(2.1)$$

where

$$I_1 = e \int_{0}^{1/e} \omega(u)\, K(eu)\, du.$$

Integrating by parts, we have

$$I_1 = e\,[O\,[uh(u)]\,K(eu)]_0^{1/e} + O[e \int_0^{1/e} uh(u)\,du\,[-\,K(eu)]\,du]$$

$$= I_1^1 + I_1^2.$$

Say, now, we obtain

$$I_1^1 = e\,[O\,[uh(u)]\,K(eu)]_0^{1/e}$$

$$= O\,[h(1/e)]$$

and

$$I_1^2 = e\,[\int_0^{1/e} \{O\,[uh(u)]\,du\,[-\,K(eu)]\}\,du]$$

$$= e\,[O\,(uh(u))\,K(eu)]_0^{1/e} + O\,[e \int_0^{1/e} \{uh'(u) + h(u)\}\,K(eu)\,du]$$

$$= O[h(1/e)] + O[e \int_0^{1/e} uh'(u)\,K(eu)\,du]$$

$$+ O[e \int_0^{1/e} h(u)\,K(eu)\,du].$$

Using condition (d) for $\beta = 1$, we have

$$\leqslant O[h(1/e)] + O[h'(1/e)]\,\mu_1 + O[h(1/e)]$$
$$\leqslant O[2h(1/e)] + O[h'(1/e)]\,\mu_1.$$

Consequently

$$I_1 \leqslant O[3h(1/e)] + O[h'(1/e)]\,\mu_1. \qquad \ldots(2.2)$$

Now, we consider I_2

$$I_2 = e \int_{1/e}^{\delta} \omega(u)\,K(eu)\,du.$$

Integrating by parts, we have

$$I_2 = e\,[O[uh(u)]\,K(eu)]_{1/e}^{\delta} + O[e \int_{1/e}^{\delta} uh(u)\,du\,[-\,K(eu)]\,du]$$

$$= I_2^1 + I_2^2.$$

Say, we obtained

$$I_2^1 = e\,[O\,(uh(u))\,K(eu)]_{1/e}^{\delta}$$

$$\leqslant O\,[h(1/e)]$$

and

$$I_2^2 = O[e \int_{1/e}^{\delta} \{uh(u)\, du(-K(eu))\}\, du]$$

$$\leqslant O[euh(u)\, K(eu)]_{1/e}^{\delta} + O[e \int_{1/e}^{\delta} \{uh'(u) + h(u)\}\, K(eu)\, du]$$

$$\leqslant O[h(1/e)] + O[e \int_{1/e}^{\delta} uh'(u)\, K(eu)\, du]$$

$$+ O[e \int_{1/e}^{\delta} h(u)\, K(eu)\, du]$$

$$\leqslant O[h(1/e)] + O[h'(1/e)\, \mu_1] + O[h(1/e)]$$

$$\leqslant O[2h(1/e)] + O[h'(1/e)]\, \mu_1.$$

Consequently

$$I_2 \leqslant O[3h(1/e)] + O[h'(1/e)]\, \mu_1. \qquad \ldots(2.3)$$

Finally, we discuss

$$|\, I_3 \,| = |\, e \int_{\delta}^{\infty} \omega(u)\, K(eu)\, du \,|$$

$$\leqslant e \int_{\delta}^{\infty} |\, \omega(u) \,|\, K(eu)\, du.$$

As $\qquad f(x) \in L_1(-\infty, \infty)$

$$|\, \omega(u) \,| \leqslant 4 \,\| f \|_{L_1} \equiv M, \text{ say}$$

therefore, we have

$$|\, I_3 \,| \leqslant Me \int_{\delta}^{\infty} K(eu)\, du$$

$$\leqslant M \sqrt{2}\pi. \qquad \ldots(2.4)$$

Combining (2.2), (2.3) and (2.4) we obtain

$$\| J_e(x) - f(x) \|_{L_1} = O[h(1/e)].$$

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