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SOME COMBINATORIAL RESULTS ON BURST CODES

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In this paper, some combinatorial results for burst codes have been proved. These are mainly extensions of the results obtained by Sharma and Dass.

INTRODUCTION

In different kinds of communication channels, one comes across with different types of errors. One of the types which occurs more frequently in many channels is that of burst errors. Moreover the study of burst errors correcting codes is now becoming more important from application point of view. Another important aspect regarding the burst codes is the study of those bursts which are of certain specified weights. This study was initiated by Sharma and Dass (1976). In this paper we propose to study results regarding the weights of bursts of length b which are of weight w or of weight w or less ($w \leqslant b$).

Consider the space of all n-tuples whose components are taken from a finite field $GF(q)$ of q code characters with elements 0, 1, 2, ..., $q - 1$. A burst of length b is considered as a vector whose only non-zero components are confined to b successive components, the first and the last of which are non-zero (Peterson 1961). The weight of a burst is taken as the Hamming weight of the vector.

COMBINATORIAL RESULTS

Let $W_{b,w}$ denote the total weight of those bursts of length b (fixed) which are of weight w or less ($w \leqslant b$) in the space of all n-tuples over $GF(q)$. Before obtaining the results we give a lemma.

Lemma — Let $[1 + x]^{(n,r)}$ denote the incomplete binomial expansion

$$1 + \binom{n}{1} x + \ldots + \binom{n}{r} x^r \text{ of } (1 + x)^n$$

up to the term x^r in the ascending powers of x.

Then $\frac{d}{dx} [1 + x]^{(n,r)} = n\,[1 + x]^{(n-1,\, r-1)}$, where $\frac{d}{dx}$ stands for the derivative with respect to x.

PROOF : We have $[1 + x]^{(n,r)} = 1 + \binom{n}{1} x + \ldots + \binom{n}{r} x^r$. This gives

$$\frac{d}{dx} [1 + x]^{(n,r)} = \binom{n}{1} + 2\binom{n}{2} x + \ldots + r\binom{n}{r} x^{r-1}$$

$$= n + \frac{2n(n-1)}{\underline{|2}} x + \ldots + \frac{rn(n-1)\ldots(n-\overline{r-1})}{\underline{|r}} x^{r-1}$$

$$= n\left[1 + \binom{n-1}{1} x + \ldots + \binom{n-1}{r-1} x^{r-1}\right]$$

$$= n\,[1 + x]^{(n-1, r-1)}.$$

Theorem 1 — In the space of all n-tuples over $GF(q)$, for

$$n \geqslant b \geqslant w > 1$$

$$W_{b,w} = (n - b + 1)(q - 1)^2 [2 [1 + (q - 1)]^{(b-2, w-2)} + (q - 1)(b - 2) [1 + (q - 1)]^{(b-3, w-3)}].$$

PROOF : We know that the total number of bursts of length $b(> 1)$ with weight w in the space of all n-tuples over $GF(q)$ is (refer to lemma of Sharma and Dass 1976), $\binom{b-2}{w-2}(n - b + 1)(q - 1)^w$. Therefore, $W_{b,w}$ the total weight of bursts of length b (fixed) with weight w or less, is given by

$$W_{b,w} = \sum_{i=2}^{w} i\binom{b-2}{i-2}(q-1)^i (n - b + 1)$$

$$= (n - b + 1)(q - 1)\left[2(q - 1) + 3\binom{b-2}{1}(q - 1)^2 + \ldots + w\binom{b-2}{w-2}(q - 1)^{w-1}\right]$$

$$= (n - b + 1)(q - 1)\frac{d}{dq}\left[(q - 1)^2 + \binom{b-2}{1}(q - 1)^3 + \ldots + \binom{b-2}{w-2}(q - 1)^w\right]$$

$$= (n - b + 1)(q - 1)\frac{d}{dq}[(q - 1)^2 [1 + (q - 1)]^{(b-2, w-2)}]$$

$$= (n - b + 1)(q - 1)^2 [2 [1 + (q - 1)]^{(b-2, w-2)} + (q - 1)(b - 2) [1 + (q - 1)]^{(b-3, w-3)}].$$

We shall now obtain two recurrence relations between

(i) $W_{b,w}$ and $W_{b-1,\,w-1}$ and (ii) $W_{b,w}$ and $W_{b,w-1}$.

Theorem 2 — A recurrence relation between $W_{b,w}$ and $W_{b-1,\,w-1}$ is given by

$$(n-b+2)(q-1)^2 \frac{d}{dq}\left[\frac{W_{b,w}}{(n-b+1)(q-1)^2}\right] = (b-2)\, W_{b-1,\,w-1}$$
$$+ (n-b+2)(b-2)(q-1)^2\,[1+(q-1)]^{(b-3,\,w-3)}.$$

PROOF : We know that

$$W_{b,w} = (n-b+1)(q-1)^2\,[2\,[1+(q-1)]^{(b-2,\,w-2)}$$
$$+ (q-1)(b-2)\,[1+(q-1)]^{(b-3,\,w-3)}].$$

Therefore,

$$W_{b-1,w-1} = (n-b+2)(q-1)^2\,\{2\,[1+(q-1)]^{(b-3,\,w-3)}$$
$$+ (q-1)(b-3)\,[1+(q-1)]^{(b-4,\,w-4)}\}$$

and

$$\frac{W_{b,w}}{(n-b+1)(q-1)^2} = 2\,[1+(q-1)]^{(b-2,\,w-2)}$$
$$+ (q-1)(b-2)\,[1+(q-1)]^{(b-3,\,w-3)}.$$

Differentiating with respect to q and using the Lemma, we get

$$\frac{d}{dq}\left[\frac{W_{b,w}}{(n-b+1)(q-1)^2}\right] = 2(b-2)\,[1+(q-1)]^{(b-3,\,w-3)}$$
$$+ (b-2)\,[\,1+(q-1)]^{(b-3,\,w-3)}$$
$$+ (q-1)(b-2)(b-3)\,[1+(q-1)]^{(b-4,\,w-4)}$$
$$= (b-2)[2\,[1+(q-1)]^{(b-3,\,w-3)} + (q-1)(b-3)\,[1+(q-1)]^{(b-4,\,w-4)}]$$
$$+ (b-2)\,[1+(q-1)]^{(b-3,\,w-3)}.$$

The result now follows by using the value of $W_{b-1,\,w-1}$.

Theorem 3 — $W_{b,w} = W_{b,w-1} + (n-b+1)\, w \binom{b-2}{w-2} (q-1)^w.$

This result can be derived by changing b to $b-1$ in the expression for $W_{b,w}$ and simplifying.

As an application of Theorem 1 we shall now obtain a bound on the largest minimum weight attainable by a burst of length b with weight w or less in the space of all n-tuples. A similar bound for obtaining the largest minimum weight attainable by a code word in an (n, k) linear code has been obtained by Plotkin (1960) [also cf. Theorem 4.1 of Peterson (1961)]. We give the bound in the next result.

Theorem 4 — The minimum weight of a burst of length b with weight w or less ($w \leqslant b$) in the space of all n-tuples over $GF(q)$ is atmost

$$2 + \frac{(q-1)(b-2)[1+(q-1)]^{(b-3,\, w-3)}}{[1+(q-1)]^{(b-2,\, w-2)}}.$$

PROOF: Using the lemma of Sharma and Dass (1976) the number of bursts of length b with weight w or less in the space of all n-tuples over $GF(q)$ is

$$(n-b+1)(q-1)^2[1+(q-1)]^{(b-2,\, w-2)}.$$

From Theorem 1, the total weight is

$$(n-b+1)(q-1)^2[2[1+(q-1)]^{(b-2,\, w-2)} + (q-1)(b-2)[1+(q-1)]^{(b-3,\, w-3)}]$$

Since the minimum weight element can have atmost average weight, the minimum weight of a burst of length b with weight w or less is atmost.

$$= \frac{(n-b+1)(q-1)^2[2[1+(q-1)]^{(b-2,\, w-2)} + (q-1)(b-2)[1+(q-1)]^{(b-3,\, w-3)}]}{(n-b+1)(q-1)^2[1+(q-1)]^{(b-2,\, w-2)}}$$

The result now follows after simplification.

Remark: It is interesting to note that the bound obtained in the preceding theorem turns out to be independent of n. Thus the bound remains the same for all n so long as $n \geqslant b > 1$.

GENERATING FUNCTION FOR $W_{b,w}$

Weight generating functions play a useful role in evaluating code words of a given weight. We state in the following theorem a result giving a generating function for $W_{b,w}$.

Theorem 5 — For $n \geqslant b \geqslant w > 1$, $W_{b,w}$ is the coefficient of x^w in the expansion of $(n-b+1)(q-1)^2 x^2 [1+(q-1)x]^{b-3}[2+(q-1)xb](1-x)^{-1}$. This result can easily be verified.

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