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ON PROJECTIVELY RELATED RECURRENT AND PROJECTIVE RECURRENT FINSLER SPACES

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This paper is devoted to the study of the effect of projective transformation on the recurrent Finsler spaces. The projective recurrence for Finsler spaces has also been defined, and some of its properties have been studied.

1. INTRODUCTION

Let F_n $(n \geqslant 2)$ be an n-dimensional Finsler space. If Berwald's curvature tensor H^i_{jkl} (Rund 1959) satisfies the relation

$$H^i_{jkl(m)} = K_m H^i_{jkl} \qquad ...(1.1)$$

where a suffix m in the brackets denotes covariant differentiation w.r.t. x^m in Berwald's sense and K_m is a non-zero covariant vector field homogeneous of degree zero in $\dot{x}^i$, we call F_n a recurrent Finsler space (Moór 1963). Let $\bar{F}_n$ be another n-dimensional Finsler space, which is obtained by a projective transformation of F_n characterized by (4.8.3) of Rund (1959). Quantities corresponding to $\bar{F}_n$ shall have bar overhead to distinguish from the quantities corresponding to F_n.

With the help of identities (4.6.21) and (4.6.22) of Rund (1959), the projective deviation tensor W^i_j defined by (4.8.10) of Rund (1959) can be written as

$$W^i_j = H^i_j - H\delta^i_j - \frac{\dot{x}^i}{n^2 - 1}\,[(n-2)\,H_{jr}\dot{x}^r - (2n-1)\,H_j]. \qquad ...(1.2)$$

Suppose F_n is a recurrent space with recurrence vector K_m. Then (1.1) holds good. On taking the covariant derivative of (1.2) and using the relations

$$H_{ij(m)} = K_m H_{ij},\ H_{j(m)} = K_m H_j,\ H^i_{j(m)} = K_m H^i_j,\ H_{(m)} = K_m H \qquad ...(1.3)$$

which are easily obtained from (1.1), we have

$$W^i_{j(m)} = K_m W^i_j \cdot \qquad ...(1.4)$$

If the projective deviation tensor W^i_j satisfies the conditions (1.4), we will call F_n as a projective recurrent Finsler space. Hence we have the following theorem.

Theorem 1.1 — A recurrent Finsler space with recurrence vector K_l is a projective recurrent Finsler space with the same recurrence vector K_l.

2. Projectively Related Recurrent Spaces

Let us assume that F_n and $\bar{F}_n$ are both recurrent with recurrence vectors K_l and $\bar{K}_l$ respectively. Then (1.1) together with the following relation holds good:

$$\bar{H}^i_{jkl/m} = \bar{K}_m H^i_{jkl} \qquad ...(2.1)$$

where oblique denotes covariant differentiation in $\bar{F}_n$ in the sense of Berwald (using the connection $\bar{G}^i_{jk}$). On taking the covariant differentiation of (4.8.7) of Rund (1959), we have

$$\bar{H}^i_{j/m} = H^i_{j/m} + \dot{x}^i \,[(\dot{\partial}_j P)_{(h)}\dot{x}^h - 2P_{(j)} - P\dot{\partial}_j P]_{/m} + \delta^i_j \,[P_{(h)}\dot{x}^h + P^2]_{/m} \qquad ...(2.2)$$

since $\dot{x}^i_{/l} = 0$. But $H^i_{j/m} = \partial_m H^i_j - \dot{\partial}_r H^i_j \,\dot{\partial}_m \bar{G}^r + H^r_j \,\bar{G}^i_{rm} - H^i_r \,\bar{G}^r_{jm}$, which on substitution from the relations (4.8.3) and (4.8.5) of Rund (1959) and taking into consideration the relations $H^i_j \,\dot{x}^j = 0$ and $\dot{\partial}_m H^i_j \,\dot{x}^m = 2H^i_j$, becomes

$$H^i_{j/m} = H^i_{j(m)} + 2H^i_j \,\dot{\partial}_m P + P\dot{\partial}_m H^i_j - \delta^i_m \,H^r_j \,\dot{\partial}_r P + H^i_m \dot{\partial}_j P - H^r_j \,\dot{\partial}_r \dot{\partial}_m P \dot{x}^i. \qquad ...(2.3)$$

On substituting (2.3) in (2.2), we have

$$\bar{H}^i_{j/m} = H^i_{j(m)} + 2H^i_j \,\dot{\partial}_m P + H^i_m \dot{\partial}_j P - \delta^i_m \,H^r_j \,\dot{\partial}_r P + P\dot{\partial}_m H^i_j - H^r_j \,\dot{\partial}_r \dot{\partial}_m P \dot{x}^i + \dot{x}^i \,[(\dot{\partial}_j P)_{(h)}\dot{x}^h - 2P_{(j)} - P\dot{\partial}_j P]_{/m} + \delta^i_j \,[P_{(h)}\dot{x}^h + P^2]_{/m}. \qquad ...(2.4)$$

From the relation (2.1) we can easily derive $\bar{H}^i_{j/m} = K_m \bar{H}^i_j$, which on substitution from (4.8.7) of Rund (1959), becomes

$$\bar{H}^i_{j/m} = \bar{K}_m [H^i_j + \dot{x}^i \{(\dot{\partial}_j P)_{(h)}\dot{x}^h - 2P_{(j)} - P\dot{\partial}_j P\} + \delta^i_j \{P_{(h)}\dot{x}^h + P^2\}]. \quad \text{...(2.5)}$$

Then, on substituting (2.5) in (2.4) and using (1.3), and rearranging, we have $(\bar{K}_m - K_m - 2\dot{\partial}_m P) H^i_j$

$$= H^i_m \dot{\partial}_j P - \delta^i_m H^r_j \dot{\partial}_r P + P\dot{\partial}_m H^i_j - H^r_j \dot{\partial}_r \dot{\partial}_m P \dot{x}^i$$
$$+ \dot{x}^i [\{(\dot{\partial}_j P)_{(h)}\dot{x}^h - 2P_{(j)} - P\dot{\partial}_j P\}_{/m} - \bar{K}_m \{(\dot{\partial}_j P)_{(h)}\dot{x}^h - 2P_{(j)}$$
$$- P\dot{\partial}_j P\}] + \delta^i_j [\{P_{(h)}\dot{x}^h + P^2\}_{/m} - \bar{K}_m \{P_{(h)}\dot{x}^h + P^2\}]. \quad \text{...(2.6)}$$

On contracting i and j in (2.6) and using $H^i_i = (n - 1) H$, $H^i_j \dot{x}^j = 0$, we have

$$(\bar{K}_m - K_m - 2\dot{\partial}_m P) - P\dot{\partial}_m H$$
$$= (P_{(h)}\dot{x}^h + P^2)_{/m} - \bar{K}_m (P_{(h)}\dot{x}^h + P^2) \quad \text{...(2.7)}$$

since $\{(\dot{\partial}_j P)_{(h)} \dot{x}^h - 2P_{(j)} - P\dot{\partial}_j P\}_{/m} \dot{x}^j = -\{P_{(h)}\dot{x}^h + P^2\}_{/m}$. On substituting (2.7) in (2.6), we have

$$(\bar{K}_m - K_m - 2\dot{\partial}_m P)(H^i_j - \delta^i_j H)$$
$$= P\dot{\partial}_m (H^i_j - \delta^i_j H) + H^i_m \dot{\partial}_j P - \delta^i_m H^r_j \dot{\partial}_r P - H^r_j \dot{\partial}_r \dot{\partial}_m P \dot{x}^i$$
$$+ \dot{x}^i [\{(\dot{\partial}_j P)_{(h)}\dot{x}^h - 2P_{(j)} - P\dot{\partial}_j P\}_{/m}$$
$$- \bar{K}_m \{(\dot{\partial}_j P)_{(h)}\dot{x}^h - 2P_{(j)} - P\dot{\partial}_j P\}]. \quad \text{...(2.8)}$$

On multiplying (2.8) by y_i $(= g_{il}\dot{x}^l)$ and rearranging, we obtain

$$\{(\dot{\partial}_j P)_{(h)}\dot{x}^h - 2P_{(j)} - P\dot{\partial}_j P\}_{/m} - \bar{K}_m \{(\dot{\partial}_j P)_{(h)}\dot{x}^h - 2P_{(j)} - P\dot{\partial}_j P\}$$
$$= F^{-1} l_r (H^r_j - \delta^r_j H)(\bar{K}_m - K_m - 2\dot{\partial}_m P) - PF^{-1} l_r \dot{\partial}_m (H^r_j - \delta^r_j H)$$
$$- F^{-1} l_r H^r_m \dot{\partial}_j P + F^{-1} l_m H^r_j \dot{\partial}_r P + H^r_j \dot{\partial}_r \dot{\partial}_m P.$$

On substituting the above relation in (2.8) and rearranging, we have

$$[(\bar{K}_m - K_m - 2\dot{\partial}_m P)(H^r_j - \delta^r_j H) - P\dot{\partial}_m (H^r_j - \delta^r_j H) - H^r_m \dot{\partial}_j P$$
$$+ \delta^r_m H^s_j \dot{\partial}_s P] \, h^i_r = 0 \quad \text{...(2.9)}$$

where $h^i_r = \delta^i_r - l^i l_r$. On multiplying (2.9) by $\dot{x}^m$ and using $H^i_j \dot{x}^j = 0$, we have $(H^r_j - \delta^r_j H) h^i_r [(\bar{K}_m - K_m)\dot{x}^m - 4P] = 0$. Since $\dot{x}^r h^i_r = 0$, the rank of the matrix (h^i_r) is $(n-1)$, and $h^i_r = 0 \Rightarrow n = 1$ (a contradiction). Thus we have the following theorem.

Theorem 2.1 — If a Finsler space F_n, which is recurrent with recurrence vector K_l, is transformed into another recurrent space $\bar{F}_n$ with recurrence vector $\bar{K}_l$ by a projective transformation characterized by (4.8.3) of Rund (1959), then at least one of the following cases occurs:

(i) $H^i_j - \delta^i_j H = \phi_j \dot{x}^i$, where ϕ_j is a covariant vector field positively homogeneous of degree one in $\dot{x}^i$ satisfying the condition det $(\phi_j \dot{x}^i) = 0$.

(ii) $(\bar{K}_l - K_l)\dot{x}^l = 4P$.

Corollary 2.1 — If a recurrent Finsler space F_n with recurrence vector K_l is transformed into another recurrent Finsler space $\bar{F}_n$ with the same recurrence vector K_l by the projective transformation characterized by (4.8.3) of Rund (1959), then at least one of the following cases occurs:

(i) $H^i_j - \delta^i_j H = \phi_j \dot{x}^i$, where ϕ_j is a covariant vector field positively homogeneous of degree one in $\dot{x}^i$ satisfying the condition det $(\phi_j \dot{x}^i) = 0$.

(ii) $P = 0$, that is, the transformation is affine.

3. Projectively Related Projective Recurrent Finsler Spaces

Let us assume that the two Finsler spaces F_n and $\bar{F}_n$ [obtained by the projective transformation of F_n characterized by (4.8.3) of Rund (1959)] are projective recurrent with K_l and $\bar{K}_l$ as the recurrence vectors. Then

$$W^i_{j(l)} = K_l W^i_j \qquad \text{...(3.1)}$$

and

$$\bar{W}^i_{j|l} = \bar{K}_l \bar{W}^i_j \text{, that is, } W^i_{j|l} = \bar{K}_l W^i_j \qquad \text{...(3.2)}$$

since $\bar{W}^i_j = W^i_j$. But

$$W^i_{j|l} = \partial_l W^i_j - \dot{\partial}_m W^i_j \, \partial_l \bar{G}^m + W^m_j \bar{G}^i_{ml} - W^i_m \bar{G}^m_{jl}$$

which, on substitution from (4.8.3) and (4.8.5) of Rund (1959), becomes

$$W^i_{j/l} = W^i_{j(l)} + P\dot{\partial}_l W^i_j + 2W^i_j \dot{\partial}_l P$$
$$- W^m_j (\delta^i_l \dot{\partial}_m P + \dot{\partial}_m \dot{\partial}_l P \dot{x}^i) + W^i_l \dot{\partial}_j P.$$

By virtue of (3.1) and (3.2), the above relation becomes

$$(\bar{K}_l - K_l) W^i_j = P\dot{\partial}_l W^i_j + 2W^i_j \dot{\partial}_l P + W^i_l \dot{\partial}_j P$$
$$- W^m_j (\delta^i_l \dot{\partial}_m P + \dot{\partial}_m \dot{\partial}_l P \dot{x}^i). \quad \ldots(3.3)$$

On contracting i and l in (3.3) and using $\dot{\partial}_l W^l_j = 0 = W^l_l$, $\dot{\partial}_m \dot{\partial}_l P \dot{x}^l = 0$, we have

$$[\bar{K}_l - K_l + (n-2) \dot{\partial}_l P] W^l_j = 0. \quad \ldots(3.4)$$

Hence, we have the following theorem.

Theorem 3.1 — If a projective recurrent Finsler space F_n with the recurrence vector K_l is transformed into another projective recurrent Finsler space $\bar{F}_n$ with recurrence vector $\bar{K}_l$ by the projective transformation characterized by (4.8.3) of Rund (1959), then at least one of the following relations holds good:

(i) $\bar{K}_l = K_l - (n-2) \dot{\partial}_l P$

(ii) $\det (W^l_j) = 0.$

Now multiplying (3.3) by $\dot{x}^l$ and using $\dot{\partial}_l W^i_j \dot{x}^l = 2W^i_j$, $\dot{\partial}_l P \dot{x}^l = P$, $W^i_l \dot{x}^l = 0$ and $\dot{\partial}_m \dot{\partial}_l P \dot{x}^l = 0$, we obtain

$$[(\bar{K}_l - K_l) \dot{x}^l - 4P] W^i_j + W^m_j \dot{\partial}_m P \dot{x}^i = 0. \quad \ldots(3.5)$$

If $\bar{K}_l = K_l$ and $n > 2$, then the relations (3.4) and (3.5) reduce to $\dot{\partial}_l P W^l_j = 0$ and $-4PW^i_j + \dot{\partial}_l P W^l_j \dot{x}^i = 0$ respectively, which implies that $PW^i_j = 0$. Conversely, if $\bar{K}_l = K_l$ and any of P or W^i_j vanishes, then the relation (3.3) is satisfied. Hence

Theorem 3.2 — A projective recurrent Finsler space F_n $(n > 2)$ with the recurrence vector K_l is transformed into another projective recurrent Finsler space $\bar{F}_n$ with the same recurrence vector K_l by the projective transformation characterized by (4.8.3) of Rund (1959), if and only if one of the following conditions is satisfied:

(i) $W^i_j = 0$, (ii) $P = 0$.

If we assume that $P \neq 0$, then we can deduce the following theorem:

Theorem 3.3 — A projective recurrent Finsler space F_n with the recurrence vector K_l, whose projective deviation tensor does not vanish identically, cannot be transformed into another projective recurrent Finsler space $\bar{F}_n$ with the same recurrence vector K_l by the projective transformation characterised by (4.8.3) of Rund (1959).

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