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ON GENERALIZATION OF BANACH CONTRACTION PRINCIPLE

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Fixed points theorems for mappings of Dass and Gupta (1975) have been obtained in arbitrary topological spaces.

Let (X, d) be a metric space. A mapping $T : X \to X$ is called a contraction mapping if there exists a positive real constant $\alpha < 1$ such that

$$d(Tx, Ty) \leqslant \alpha d(x, y), \; \forall \; x, y \in X.$$

By the well-known Banach's contraction principle every contraction mapping of a complete metric space (X, d) into itself has a unique fixed point.

Recently Dass and Gupta (1975) generalized Banach's contraction principle for mappings T satisfying

$$d(Tx, Ty) \leqslant \alpha d(y, Ty) \frac{[1 + d(x, Tx)]}{1 + d(x, y)} + \beta d(x, y) \qquad \ldots(1)$$

for all $x, y \in X$ and for $\alpha, \beta > 0$, $\alpha + \beta < 1$.

The object of this note is to prove some fixed point theorems in arbitrary topological spaces. We prove the following theorems.

Theorem 1 — Let T be a continuous mapping of a Hausdorff space X into itself and let $F : X \times X \to [0, \alpha)$ be a continuous mapping such that for each pair of distinct points $x, y \in X$,

$$F(Tx, Ty) \leqslant \frac{\alpha F(y, Ty) [1 + F(x, Tx)]}{1 + F(x, y)} + \beta(x, y) \qquad \ldots(2)$$

where $\alpha, \beta \geqslant 0$ are constants such that $\alpha + \beta < 1$.

If for some $x_0 \in X$, the sequence of iterates $\{T^n x_0\}$ has a subsequence $\{T^{n_k} x_0\}$ converging to $z \in X$, then z is a fixed point of T.

PROOF : We have the monotone sequence of non-negative real numbers

$$F(x_0, Tx_0) > F(Tx_0, T^2x_0) > \ldots > F(T^n x_0, T^{n+1} x_0) > \ldots$$

which must converge along with all its subsequences to some real number λ.

Now from the continuity of F and T we have,

$$
\begin{aligned}
F(z, Tz) &= F(\lim_{k\to\infty} T^{n_k} x_0, T \lim_{k\to\infty} T^{n_k} x_0) \\
&= F(\lim_{k\to\infty} T^{n_k} x_0, \lim_{k\to\infty} T^{n_k+1} x_0) \\
&= \lim_{k\to\infty} F(T^{n_k} x_0, T^{n_k+1} x_0) \\
&= \lim_{k\to\infty} F(T^{n_k+1} x_0, T^{n_k+2} x_0) \\
&= F(\lim_{k\to\infty} T^{n_k+1} x_0, \lim_{k\to\infty} T^{n_k+2} x_0) = F(Tz, T^2 z).
\end{aligned}
$$

If $z \neq Tz$, then from (2)

$$F(Tz, T^2z) \leqslant \frac{\alpha F(Tz, T^2z)\,[1 + F(z, Tz)]}{1 + F(z, Tz)} + \beta F(z, Tz)$$

which gives $F(z, Tz) = F(Tz, T^2z) < F(z, Tz)$, a contradiction.

Thus z is a fixed point of F.

Corollary 1 — Let X be a metric space with metric d and let T be a continuous mapping of X into itself such that

$$d(Tx, Ty) \leqslant \frac{\alpha d(y, Ty)\,[1 + d(x, Tx)]}{1 + d(x, y)} + \beta d(x, y)$$

for every pair of elements $x, y \in X$, and for constants $\alpha, \beta \geqslant 0$ satisfying $\alpha + \beta < 1$. If for some $x_0 \in X$ the sequence of iterates $\{T^n x_0\}$ has a subsequence $\{T^{n_k} x_0\}$ converging to some $z \in X$, then z is a unique fixed point of T.

PROOF : The proof follows from Theorem 1 by taking $d = F$.

Corollary 2 — The result of Edelstein (1962, Theorem 1) follows by taking $\alpha = 0$.

Theorem 2 — Let (X, d) be a complete metric space and let $T : X \to X$ satisfying

$$
\begin{aligned}
d(T^{p+1}x, T^{p+2}y) \leqslant{} & \frac{\alpha d(T^{p+1}y, T^{p+2}y)\,[1 + d(T^p x, T^{p+1}x)]}{1 + d(T^p x, T^{p+1} y)} \\
& + \beta d(T^p x, T^{p+1} y) \qquad \ldots(3)
\end{aligned}
$$

$\forall\ x, y \in X$ and for non-negative constants α, β with $\alpha + \beta < 1$, and for any non-negative integer p.

Then T has a unique fixed point.

PROOF: We prove the theorem for $p = 0$. The proof in the general case follows on similar lines.

For $p = 0$ we have from (3)

$$d(Tx, T^2y) \leqslant \frac{\alpha d(Ty, T^2y)[1 + d(x, Tx)]}{1 + d(x, Ty)} + \beta d(x, Ty).$$

We define a sequence of elements $\{x_n\} \in X$ as follows:

$$x_n = T^n x_0,\ x_{n+1} = Tx_n,\ n = 0, 1, 2, \ldots,\ x_0 \in X \text{ being arbitrary.}$$

Then $\quad d(x_1, x_2) = d(Tx_0, T^2x_0)$

$$\leqslant \alpha d(Tx_0, T^2x_0)\frac{[1 + d(x_0, Tx_0)]}{1 + d(x_0, Tx_0)} + \beta d(x_0, Tx_0)$$

i.e. $\quad d(x_1, x_2) \leqslant \frac{\beta}{1-\alpha} d(x_0, Tx_0).$

Similarly $\quad d(x_2, x_3) = d(Tx_1, T^2x_1)$

$$\leqslant \frac{\alpha d(Tx_1, T^2x_1)[1 + d(x_1, Tx_1)]}{1 + d(x_1, Tx_1)} + \beta d(x_1, Tx_1)$$

i.e. $\quad d(x_2, x_3) \leqslant \frac{\beta}{1-\alpha} d(x_1, x_2) \leqslant \left(\frac{\beta}{1-\alpha}\right)^2 d(x_0, x_1)$

and in general

$$d(x_n, x_{n+1}) \leqslant \left(\frac{\beta}{1-\alpha}\right)^n d(x_0, x_1).$$

Since $\alpha + \beta < 1$, it follows that $\{x_n\}$ is a Cauchy sequence and from the completeness of X it follows that $\lim_{n\to\infty} x_n = \xi \in X$.

Now $\quad d(\xi, T\xi) \leqslant d(\xi, x_n) + d(x_n, T\xi)$

$$= d(\xi, x_n) + d(Tx_{n-1}, T\xi)$$

$$= d(\xi, x_n) + d(T\xi, T^2x_{n-2})$$

$$\leqslant d(\xi, x_n) + \frac{\alpha d(Tx_{n-2}) T^2x_{n-2}) [1 + d(\xi, T\xi)]}{1 + d(\xi, Tx_{n-2})} + \beta d(\xi, Tx_{n-2})$$

$$= d(\xi, x_n) + \frac{\alpha d(x_{n-1}, x_n) [1 + d(\xi, T\xi)]}{1 + d(\xi, x_{n-1})} + \beta d(\xi, x_{n-1}).$$

Taking the limit $n \to \infty$ we get, $d(\xi, T\xi) = 0$ i.e. $\xi = T\xi$.

Finally, let $\eta \in X$ be such that $\eta = T\eta$. Then

$$d(\xi, \eta) = d(T\xi, T^2\eta)$$

$$\leqslant \alpha d(T\eta, T^2\eta) \frac{[1 + d(\xi, T\xi)]}{1 + d(\xi, T\eta)} + \beta d(\xi, T\eta)$$

i.e. $(1 - \beta)\, d(\xi, \eta) \leqslant 0$, which gives $d(\xi, \eta) = 0$ i.e. $\xi = \eta$ showing the uniqueness of ξ. This completes the proof of the theorem.

We illustrate Theorem 1 by the following example.

Example : Let $X = [0, 1]$ with the usual metric and $T : X \to X$ be defined as

$$Tx = 0,\ x \neq \tfrac{1}{3}$$

$$T(\tfrac{1}{3}) = 1.$$

Clearly, T is not continuous and therefore does not satisfy $d(Tx, Ty) \leqslant \alpha d(x, y)$. But by taking $x = \frac{1}{2}$ and $y = 0$ we see that T does not satisfy (1). However, T satisfies (3) with $p = 1$.

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