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UNITARY SCHEME MODEL STUDY OF ^3H WITH GOGNY, PIRES AND DE TOURREIL INTERACTION

S. B. DOMA AND M. K. YOUSSEF

Mathematics Department, Faculty of Science, Alexandria University, Alexandria, Egypt

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The basis functions of the unitary scheme model are used to calculate the nuclear wave function, the binding energy, the charged nuclear radius and the magnetic dipole moment of the triton nucleus. Gogny, Pires and De Tourreil potential is used for the two-nucleon interaction. The calculated value of the binding energy of triton is estimated to be 8.06 MeV, that of the charged nuclear radius is 1.788 fm and they are in good agreement with the experimental values. The calculated value of the magnetic dipole moment is 3.74 N.M. which is not in a good agreement with the experimental one.

1. INTRODUCTION

The unitary scheme model (USM) (Vanagas 1971) has shown good results for the structure of light nuclei. This model considers the nucleus as a system of noninteracting quasi-particles and enables us to apply the algebraic methods for studying the general features of matrix elements of operators that correspond to physical quantities. The basis functions of this model are constructed in such a way that they will have certain symmetry with respect to the interchange of particles and have definite total angular momentum.

In this paper, we shall apply the USM to calculate the ground-state wave function, the binding energy, the charged nuclear radius, and the magnetic dipole moment of triton. The interaction used in this calculation, GPT potential (Gogny *et al.* 1970), is a two-body interaction with central, tensor, spin-orbit, and square spin-orbit potentials and gives an acceptable fit to two nucleon data up to 300 MeV.

The ground-state wave function for triton is expanded in terms of the USM basis functions with number of quanta of excitations $0 \leqslant N \leqslant 8$. In principle, the predicted results for the nuclear characteristics should be independent of the particular bases chosen (model-independent results) when the number of terms in the expansion is kept large enough. Jackson and Elliott (1969) have obtained, nearly, model-independent result for the triton binding energy but with shell-model calculations and their result was not in good agreement with the experimental one. Macharadze and Mikhelashvili (1971) have also obtained model-independent result for the binding energy of ^3H in the USM using nucleon-nucleon interaction in the form of a square-well potential but, unfortunately, their result was an over bound one. Furthermore Doma (1978) has also obtained model-independent results for the binding energy and the charged nuclear radius of ^3H in the USM using effective interactions of symmetric and Serber exchange-force potentials.

The results obtained in this paper with GPT potential are, as would be expected, more adequate and in better agreement with the experimental findings than that with the effective interactions.

2. Construction of the Wave Function and the Energy Matrix

By adding and subtracting an oscillator potential referred to the nucleus centre of mass, the internal Hamiltonian of A nucleons may then be written as:

$$H = \frac{1}{2m} \sum_{i=1}^{A} P_i^2 - \frac{1}{2mA} \left(\sum_{i=1}^{A} \vec{P}_i \right)^2 + \tfrac{1}{2} m\omega^2 \times \sum_{i=1}^{A} \left(\vec{r}_i - \frac{1}{A} \sum_{t=1}^{A} \vec{r}_t \right)^2 + \sum_{i<j=1}^{A} V(|\vec{r}_i - \vec{r}_j|) - \tfrac{1}{2} m\omega^2 \sum_{i=1}^{A} \left(\vec{r}_i - \frac{1}{A} \sum_{t=1}^{A} \vec{r}_t \right)^2 . \quad \text{...(2.1)}$$

In terms of the relative coordinates of the A nucleons such a Hamiltonian is considered to be of the form (Doma *et al.* 1975)

$$H = H^{(0)} + V' \quad \text{...(2.2)}$$

where

$$H^{(0)} = \frac{1}{A} \sum_{i<j=1}^{A} \left[\frac{1}{2m} (\vec{p}_i - \vec{p}_j)^2 + \frac{m\omega^2}{2} (\vec{r}_i - \vec{r}_j)^2 \right] \quad \text{...(2.3)}$$

is the USM Hamiltonian, and

$$V' = \sum_{i<j=1}^{A} \left[V(|\vec{r}_i - \vec{r}_j|) - \frac{m\omega^2}{2A} (\vec{r}_i - \vec{r}_j)^2 \right] \quad \text{...(2.4)}$$

is the residual interaction.

The USM basis functions, with the usual notations, have the form (Doma *et al.* 1975).

$$| A\,\Gamma M_L;\, \Gamma_S M_S M_T \rangle \equiv | AN\{\rho\}^{\alpha L M_L}_{(\nu)[f]};\, SM_S TM_T \rangle \qquad ...(2.5)$$

where Γ, and Γ_S are the sets of all orbital, and spin-isospin quantum numbers characterizing the states, respectively. By virtue of the last relation, let us construct basis functions with given total momentum J, in the usual manner as

$$\left| A\,\Gamma SJM_J TM_T \right\rangle = \sum_{M_L+M_S=M_J} (LM_L,\, SM_S \mid JM_J))\, A\,\Gamma M_L;\, \Gamma_S M_S M_T \rangle \qquad ...(2.6)$$

where $(LM_L,\, SM_S \mid JM_J)$ is the Clebsh-Gordan coefficient of the rotational group R_3. The eigenfunctions which are given by eqn. (2.6), belong to the Hamiltonian operator $H^{(0)}$ and are associated with the eigenvalues (Doma *et al.* 1975)

$$E_N^{(0)} = [N + \tfrac{3}{2}(A - 1)]\,\hbar\omega. \qquad ...(2.7)$$

The nuclear wave function is expanded in terms of the USM basis functions as follows

$$\left| J^\pi TM_J M_T \right\rangle = \sum_{\Gamma,S} C^{J^\pi T}_{\Gamma,S} \left| A\,\Gamma SJM_J TM_T \right\rangle \qquad ...(2.8)$$

where $C^{J^\pi T}_{\Gamma,S}$ are the state-expansion coefficients, and π defines the parity of the state. In the summation occurring in eqn. (2.8) N is permitted to be either even or odd integer depending on the parity.

The nuclear wave function as given by eqn. (2.8) is an eigenfunction of the operator V'. The matrix elements of this operator have the form (Doma *et al.* 1975)

$$\langle J^\pi TM_J M_T \mid V' \mid J^\pi TM_J M_T \rangle = \frac{A(A-1)}{2} \sum_{\Gamma S \bar{\Gamma} \bar{S} \bar{J} \Gamma_a \Gamma' S' \Gamma_a}$$

$$\times\, C^{J^\pi T}_{\Gamma S}\, C^{J^\pi T}_{\Gamma' S'} \langle A\,\Gamma SJT \mid A - 2\,\bar{\Gamma}\bar{S}\bar{J}\bar{T};\, 2\,\Gamma_a \rangle$$

$$\times \langle A\,\Gamma' S' JT \mid A - 2\bar{\Gamma}\bar{S}\bar{J}\bar{T};\, 2\Gamma'_a \rangle \begin{pmatrix} \bar{L} & \bar{S} & \bar{J} \\ l & s & j \\ L & S & J \end{pmatrix} \begin{pmatrix} \bar{L} & \bar{S} & \bar{J} \\ l' & s & j \\ L' & S' & J \end{pmatrix}$$

$$\times \langle (\epsilon ls)\, jt \mid V'_{A-1,A} \mid (\epsilon' l' s)\, jt \rangle, \qquad ...(2.9)$$

where $< A\,\Gamma SJT \mid A - 2\bar{\Gamma}\bar{S}\bar{J}\bar{T};\, 2\Gamma_a >$ stand for the two-particle total fractional parentage coefficients (Doma and Machabeli 1975, Doma 1975) (which are products of orbital and spin-isospin coefficients), the base is referred to the set of $A - 2$

nucleons, Γ_a is the set of all orbital and spin-isospin quantum numbers characterizing the two-particle states, and $\epsilon = 2n + l$, in which n is the radial quantum number of the inter-particle distance joining the last pair. In eqn. (2.9) $\begin{pmatrix} j_1 & j_2 & j_{12} \\ j_3 & j_4 & j_{34} \\ j_{13} & j_{24} & J \end{pmatrix}$ are the normalized 9j-symbols, and $< (\epsilon ls)\, jt \mid V'_{A-1,A} \mid (\epsilon' l' s)\, jt >$ are the two-particle matrix elements of the operator V' which are given by

$$\begin{aligned} & \langle (\epsilon ls)\, jt \mid V'_{A-1,A} \mid (\epsilon' l' s)\, jt \rangle \\ &= \langle (\epsilon ls)\, jt \mid V(\mid \vec{r}_{A-1} - \vec{r}_A \mid) \mid (\epsilon' l' s)\, jt \rangle \\ &- \langle \epsilon l \left| \frac{m\omega^2}{2A} (\vec{r}_{A-1} - \vec{r}_A)^2 \right| \epsilon' l' \rangle. \end{aligned} \qquad \text{...(2.10)}$$

For the interaction $V(\mid \vec{r}_{A-1} - \vec{r}_A \mid)$ the GPT potential (Gogny 1970) has been used which is defined as

$$V(r) = V_c(r) + V_T(r)\, S_{12} + V_{LS}(r)\, \vec{l} \cdot \vec{s} + V_{LL}(r)\, L_{12}$$

where $V_c(r)$, $V_T(r)$, $V_{LS}(r)$, and $V_{LL}(r)$ are the central, tensor, spin-orbit, and square spin-orbit interaction parts. The operator L_{12} depends upon the two-particle spins and the orbital-angular momentum corresponding to the inter-particle coordinates as follows:

$$L_{12} = (\vec{\sigma}_1 \cdot \vec{\sigma}_2)\, l^2 - \tfrac{1}{2} \{ (\vec{\sigma}_1 \cdot \vec{l}\,)(\vec{\sigma}_2 \cdot \vec{l}\,) + (\vec{\sigma}_2 \cdot \vec{l}\,)(\vec{\sigma}_1 \cdot \vec{l}\,) \}$$

and each of the radial functions $V_i(r)$ ($i = C, T, LS, LL$) is expressed as a sum of Gaussian potentials:

$$V_i(r) = \sum_{\beta} V_\beta \exp(- r^2/r_\beta^2)$$

where V_β, and r_β are the strengths, and the ranges of the potentials respectively.

3. The Charged Nuclear Radius and the Magnetic Dipole Moment

The charged nuclear radius is defined as

$$\mathcal{R} = \sqrt{r_p^2 + \langle R^2_{Nuc} \rangle} \qquad \text{...(3.1)}$$

where $r_p = 0.85$ fm is the proton radius, and the second term is the mean value of the operator (Doma 1978)

$$R^2_{Nuc} = \frac{1}{A^2} \sum_{i<j=1}^{A} r_{ij}^2. \qquad \text{...(3.2)}$$

Since this operator does not depend on the spin-isospin variables of the nuclear wave function and from the orthonormality conditions of the spin-isospin fractional parentage coefficients one can obtain (Doma 1978)

$$\mathcal{R} = \Bigg[r_p^2 + \frac{A-1}{A\left[\frac{M\omega}{\hbar}\right]} \sum_{\Gamma\Gamma'\bar{\Gamma}\Gamma_{0a}\Gamma'_{0a}} C^{J^\pi T}_{\Gamma S} C^{J^\pi T}_{\Gamma' S}$$

$$\times < A\Gamma \mid A - 2\bar{\Gamma}, 2\Gamma_{0a} > < A\Gamma' \mid A - 2\bar{\Gamma}, 2\Gamma'_{0a} > \{(\epsilon + \tfrac{3}{2})\, \delta^{\epsilon'}_{\epsilon}$$

$$- \tfrac{1}{2}\sqrt{(\epsilon - l + 2)(\epsilon + l + 3)}\, \delta^{\epsilon'}_{\epsilon+2} - \tfrac{1}{2}\sqrt{(\epsilon - l)(\epsilon + l + 1)}\, \delta^{\epsilon'}_{\epsilon-2}\}$$

$$\times \delta^{(\nu')[f']}_{(\nu)[f]} \Bigg]^{1/2} \qquad \ldots(3.3)$$

where $[M\omega/\hbar]$, Γ_{0a}, and $< A\Gamma \mid A - 2\bar{\Gamma}; 2\Gamma_{0a} >$ denote the value of $M\omega/\hbar$, the set of all orbital quantum numbers of the two-particle states, and the two-particle orbital fractional parentage coefficients, respectively.

The magnetic dipole moment μ of a nucleus is defined as the mean value of the operator $\hat{\mu} = \mu_\sigma + \mu_0$, where (Vanagas 1971)

$$\mu_\sigma = \sum_{i=1}^{A} [(\mu_p + \mu_n) + 2(\mu_p - \mu_n)\, t_{0i}]\, s_{0i} \qquad \ldots(3.4)$$

and

$$\mu_o = \tfrac{1}{2} \sum_{i=1}^{A} (1 - 2t_{0i})\, l_{0i} \qquad \ldots(3.5)$$

calculated in a state with $M_J = J$. In eqns. (3.4), and (3.5) μ_p, and μ_n are the proton, and the neutron magnetic moments respectively, t_{0i}, s_{0i}, and l_{0i} are the z-components of the isospin, spin, and orbital momenta of the ith nucleon. Writing each of the two operators μ_σ, and μ_0 as a sum of symmetric and antisymmetric operators of symmetry-types $[A]$ and $[A-1, 1]$

$$\mu_\sigma = \mu_\sigma^{[A]} + \mu_\sigma^{[A-1,1]}, \quad \mu_0 = \mu_0^{[A]} + \mu_0^{[A-1,1]}$$

the mean value of the magnetic dipole moment can be transformed to an algebraic expression depending on the orbital and the spin-isospin quantum numbers of the A-nucleon states, and the calculations are, thus, straightforward (Vanagas 1971).

4. Results and Conclusions

To calculate the binding energy of ^3H (BET) in accordance with the wave function given by eqn. (2.8) one sets $A = 3$, $J = \frac{1}{2}$, $T = \frac{1}{2}$ and consider the case of even numbers of quanta of excitations N, in the range $0 \leqslant N \leqslant 8$ (even parity states). The triton energy matrix, constructed according to eqns. (2.7) and (2.10), is a function of the oscillator parameter $\hbar\omega$ occurring in the wave function. Diagonalizing this matrix one gets the energy eigenvalues and the corresponding eigenfunctions for every assigning value of the parameter $\hbar\omega$ which is allowed to vary in the energy range $10 \leqslant \hbar\omega \leqslant 15$.

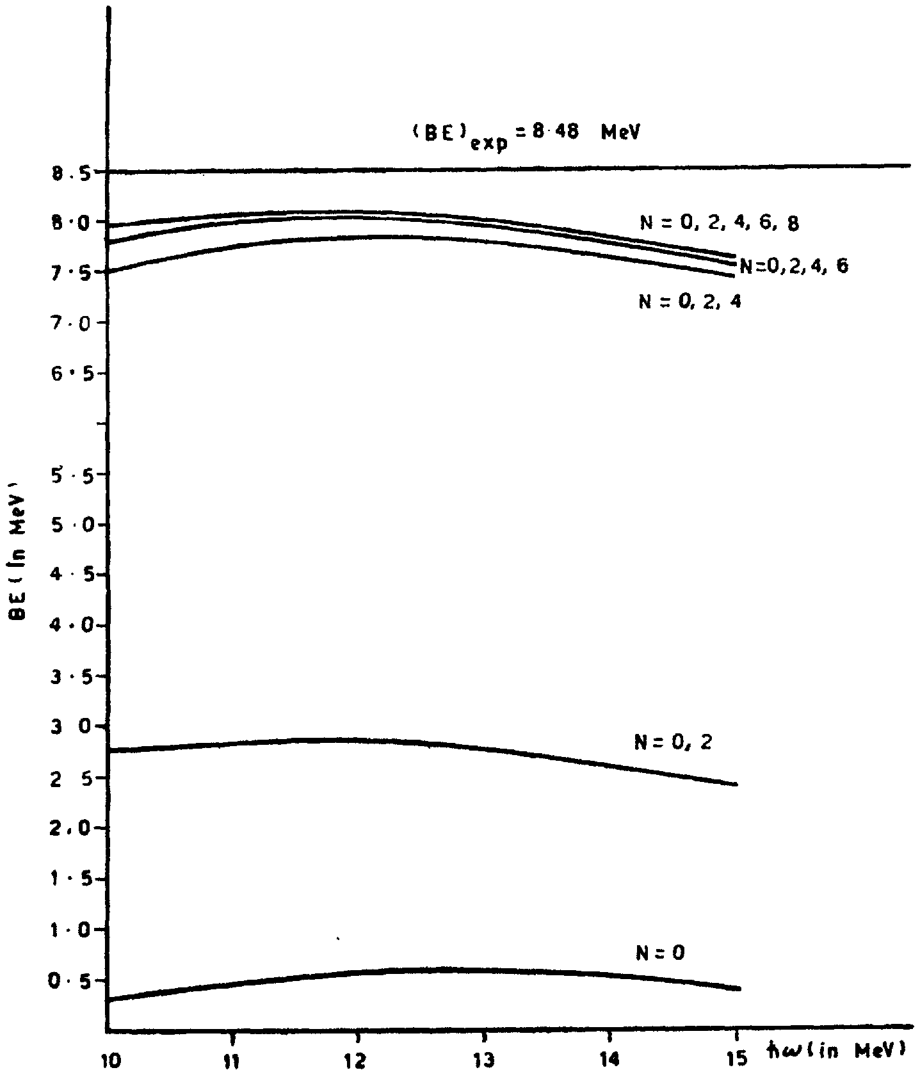

Fig. 1.

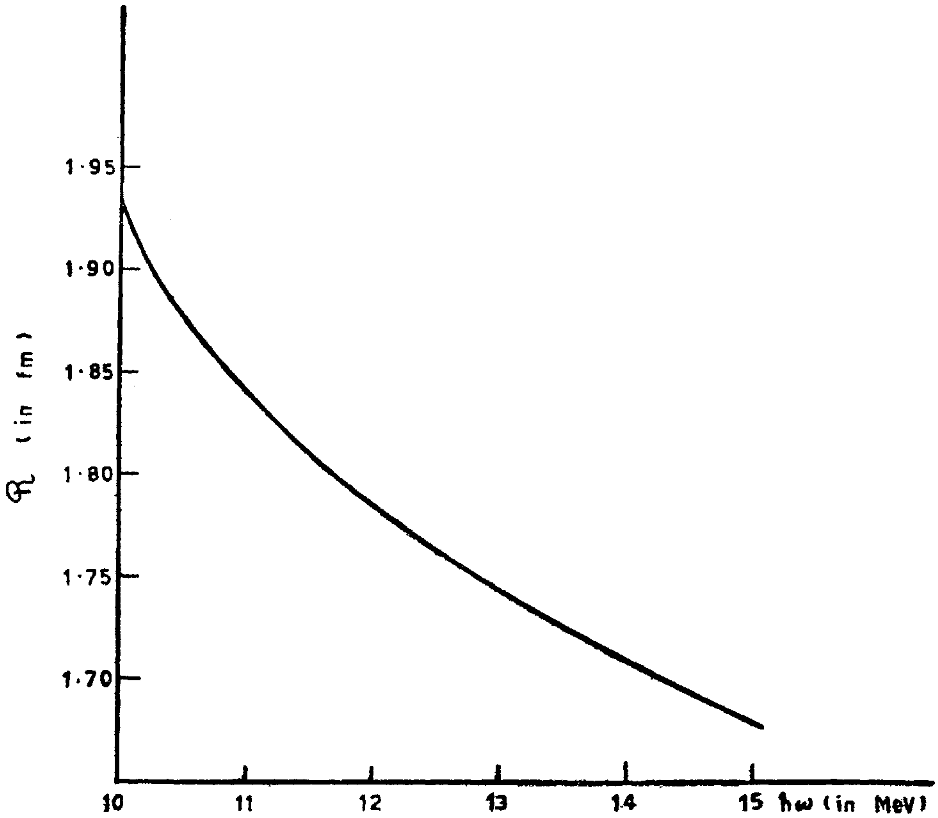

FIG. 2.

The calculated BET is estimated to be 8.06 MeV at $\hbar\omega = 12$ MeV which is about 95% of the experimental value 8.48 MeV. The corresponding nuclear wave function for ^3H is expressed in terms of the USM basis functions $| N \{\rho\} (\nu) [f] LS >$ as follows

$$\begin{aligned}\psi = {} & 0.8466 \mid 0 \{0\} (0) [3] 0\tfrac{1}{2} \rangle + 0.0929 \mid 2 \{2\} (0) [3] 0\tfrac{1}{2} \rangle \\ & - 0.3519 \mid 2 \{2\} (2) [21] 0\tfrac{1}{2} \rangle - 0.1267 \mid 2 \{2\} (2) [21] 2\tfrac{3}{2} \rangle \\ & - .0021 \mid 2 \{11\} (0)^* [111] 1\tfrac{1}{2} \rangle + 0.1781 \mid 4 \{4\} (0) [3] 0\tfrac{1}{2} \rangle \\ & + 0.1829 \mid 4 \{4\} (2) [21] 0\tfrac{1}{2} \rangle + 0.2563 \mid 4 \{4\} (2) [21] 2\tfrac{3}{2} \rangle \\ & + 0.0085 \mid 4 \{31\} (2) [21] 1\tfrac{1}{2} \rangle + 0.0227 \mid 6 \{6\} (0) [3] 0\tfrac{1}{2} \rangle \\ & - 0.0464 \mid 6 \{6\} (2) [21] 0\tfrac{1}{2} \rangle + 0.0317 \mid 8 \{8\} (0) [3] 0\tfrac{1}{2} \rangle \\ & + \text{terms of expansion coefficients} \leqslant 10^{-3}. \qquad \text{...(4.1)}\end{aligned}$$

The triton wave function given by eqn. (4.1) is used to calculate the charged nuclear radius $\mathcal{R}$ and the magnetic dipole moment μ and the results are as follows:

$$\mathcal{R} = 1.788 \text{ fm, and } \mu = 3.74 \text{ N.M.}$$

In Fig. 1 we present the variation of the BET with the oscillator parameter $\hbar\omega$ for the different values of N. Variation of $\mathcal{R}$, and μ with the oscillator parameter $\hbar\omega$ are represented in Figs. 2 and 3. Figures 4, 5, and 6 illustrate the dependence of the BET, $\mathcal{R}$ and μ upon the number of quanta of excitations N.

As one can see the obtained values of the BET, and the charged nuclear radius of ^3H are in a very good agreement with the experimental findings. Also from the results one can see that the calculated value of the magnetic dipole moment is not in good agreement with the experimental value 2.98 N.M. It is of interest to notice that the inclusion of high configurations corresponding to $N > 0$ with the ground configuration $N = 0$ is very important if one requires to obtain best fit between the calculated values for the different characteristics of nuclei and the experimental ones. One also can notice that for a small number of terms in the total nuclear wave function, the obtained results depend mainly on the oscillator parameter $\hbar\omega$ (the

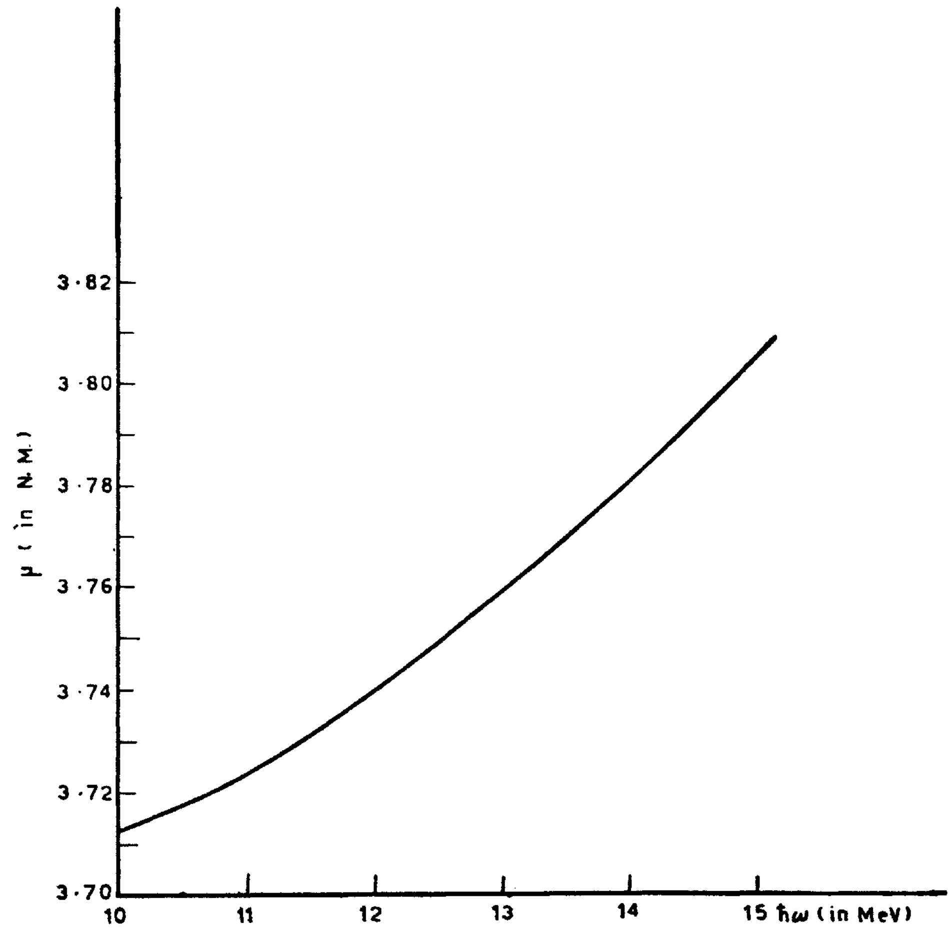

FIG. 3.

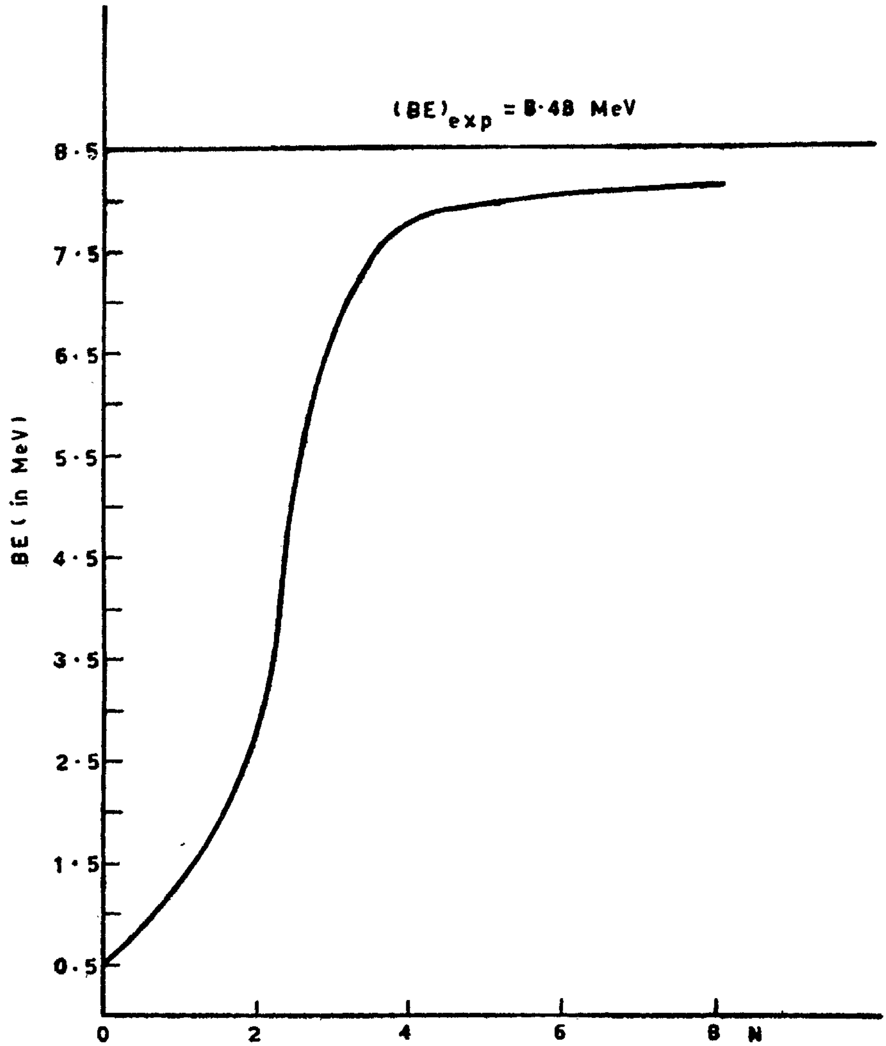

FIG. 4.

model parameter). The role of $\hbar\omega$ in calculating BET becomes negligible and tends to the extreme case of model-independent calculations, for good choice of enough number of terms in the total nuclear wave function. Finally it is of interest to notice that the contribution of states with symmetry type [21] is not to be neglected in the nuclear wave function (about 24% of the total weight).

The results obtained in this paper concerning BET and $\mathcal{R}$ are in accordance with the experimental findings and are much better than the results obtained by others (Akaishi *et al.* 1974, Jackson and Elliott 1969, Macharadze and Mikhelashvili 1971, Demin and Efross 1972). This proves that the mathematical treatments used are reliable.

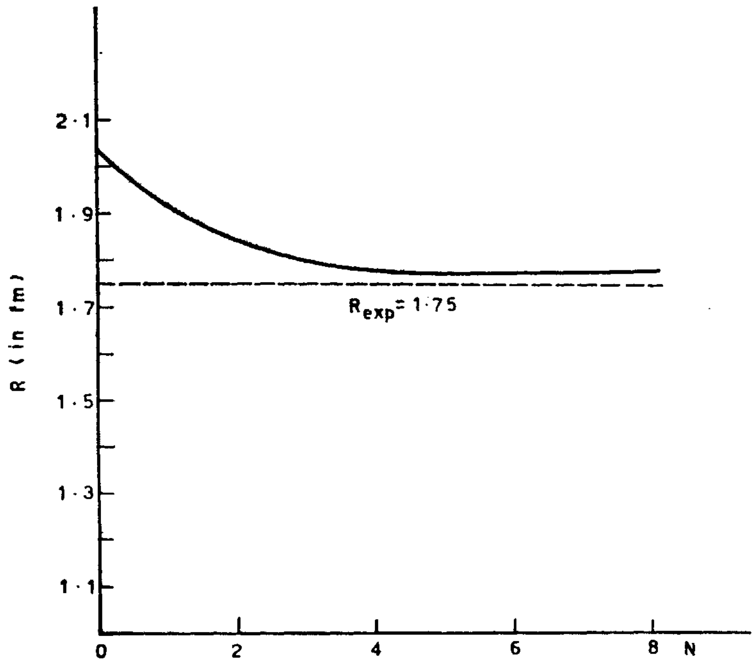
2·1
1·9
1·7
1·5
1·3
1·1
R (in fm)
R_{exp}= 1·75
0
2
4
6
8
N

FIG. 5.

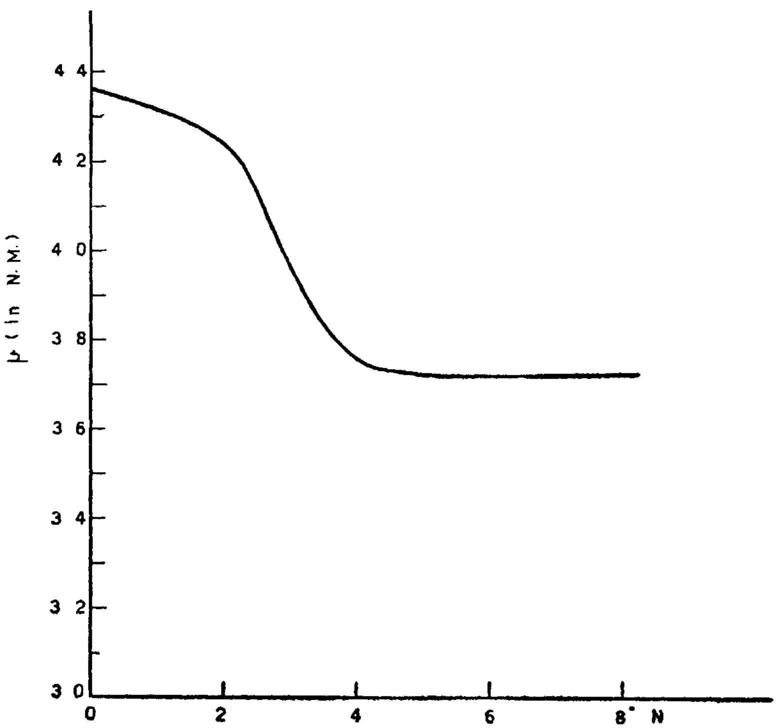
4 4
4 2
4 0
3 8
3 6
3 4
3 2
3 0
μ (in N.M.)
0
2
4
6
8
N

FIG. 6.

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