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ON COMMON FIXED POINT THEOREMS OF CONTINUOUS MAPPINGS*

CHEH-CHIH YEH

Department of Mathematics, Central University, Chung-Li, Taiwan 320

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Let E, F and T be mappings of a complete metric space (X, d) satisfying

$$ET = TE, \; FT = TF, \; E(X) \subset T(X), \; F(X) \subset T(X)$$

and

$$d(Ex, Fy) \leqslant h(d(Tx, Ty), \, d(Tx, Ex), \, d(Tx, Fy), \, d(Ty, Ex), \, d(Ty, Fy))$$

where h is upper semicontinuous and nondecreasing with respect to each variable and $h(t, t, at, bt, t) < t$. Under these assumptions our main results states that E, F and T have a unique fixed point.

1. INTRODUCTION

The results of this note are inspired by a recent paper of Jungck (1976a). He proved the following interesting result :

Theorem 0 — Let S and T be two continuous and commuting selfmappings of a complete metric space (X, d) satisfying: (a) $S(X) \subset T(X)$; (b) $d(Sx, Sy) \leqslant pd(Tx, Ty)$ for all x, y in X, where p is a nonnegative real number, $p < 1$.

Then S and T have a unique common fixed point.

In this paper, we shall extend his result to more general case. We also apply it to obtain some fixed point theorems. For related results, we refer to Husain and Sehgal (1975), Iseki (1974, 1975), Jungck (1976b), Murakami and Yeh (1978), Singh and Meade (1977) and Yeh (1978).

Let R^+ denote the set of nonnegative real numbers. Let H denote a family of mappings such that for each h in H, $h : (R^+)^5 \to R^+$ and h is upper semicontinuous and nondecreasing in each coordinate variable.

In order to obtain our main result, we need the following lemma which is due to Matkowski (1977) and Singh and Meade (1977).

Lemma — Suppose that f is a nondecreasing upper semicontinuous mapping of R^+ into itself. Then for every $t > 0$, $f(t) < t$ if and only if $\lim_{n\to\infty} f^n(t) = 0$, where f^n denotes the composition of f with itself n times.

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2. Main Results

Theorem 1 — Let E, F and T be three continuous selfmappings of a complete metric space (X, d) satisfying the following conditions :

(C_1) $ET = TE$, $FT = TF$, $E(X) \subset T(X)$ and $F(X) \subset T(X)$;

(C_2) there exists an h in H such that for all x, y in X,

$d(Ex, Fy) \leqslant h(d(Tx, Ty), d(Tx, Ex), d(Tx, Fy), d(Ty, Ex), d(Ty, Fy))$

where h satisfies the condition :

(C_3) $g(t) \equiv h(t, t, at, bt, t) < t$

for each t in $R^+ - \{0\}$, where $a + b = 2$.

Then E, F and T have a unique common fixed point in X.

Proof : Let x_0 be any point in X. Let x_1 in X be such that $Tx_1 = Ex_0$ and x_2 in X such that $Tx_2 = Fx_1$. In general, one can choose x_{2n+1} and x_{2n+2} such that

$$Tx_{2n+1} = Ex_{2n}, \quad Tx_{2n+2} = Fx_{2n+1} \qquad \ldots(1)$$

for $n = 0, 1, \ldots$. We can do this since (C_1) holds.

Let $d_n = d(Tx_n, Tx_{n+1})$ for $n = 0, 1, \ldots$. We prove that

$$d_{2n+1} \leqslant d_{2n}. \qquad \ldots(2)$$

Assume that $t \equiv d_{2n+1} > d_{2n}$. If $t > 0$, then by (C_2),

$$\begin{aligned} t &= d(Ex_{2n}, Fx_{2n+1}) \\ &\leqslant h(d(Tx_{2n}, Tx_{2n+1}), d(Tx_{2n}, Tx_{2n+1}), d(Tx_{2n}, Tx_{2n+2}), \\ &\qquad d(Tx_{2n+1}, Tx_{2n+1}), d(Tx_{2n+1}, Tx_{2n+2})) \\ &\leqslant h(t, t, 2t, t, t) < t, \end{aligned}$$

a contradiction. This contradiction proves (2) for $n = 0, 1, \ldots$. Similarly, we can prove $d_{2n+2} \leqslant d_{2n+1}$ for $n = 0, 1, \ldots$. Thus $\{d_n\}_{n=0}^{\infty}$ is decreasing. Hence

$$\begin{aligned} d_1 &= d(Tx_1, Tx_2) = d(Ex_0, Fx_1) \\ &\leqslant h(d(Tx_0, Tx_1), d(Tx_0, Ex_0), d(Tx_0, Fx_1), d(Tx_1, Ex_0), d(Tx_1, Fx_1)) \\ &\leqslant h(d_0, d_0, 2d_0, d_0, d_0) = g(d_0). \end{aligned}$$

In general, $d_n \leqslant g^n(d_0)$. If $d_0 > 0$, then, by Lemma,

$$\lim_{n \to \infty} d_n = 0. \qquad \ldots(3)$$

If $d_0 = 0$, then $d_n = 0$ for each n. Hence (3) also holds.

In order to show that $\{Tx_n\}$ is a Cauchy sequence, it is sufficient to show that $\{Tx_{2n}\}$ is a Cauchy sequence. Suppose that $\{Tx_{2n}\}$ is not a Cauchy sequence. Then there is an $\epsilon > 0$ such that for each even integer $2k$, there are even integers $2m(k)$, $2n(k)$ such that

$$d(Tx_{2m(k)}, Tx_{2n(k)}) > \epsilon \qquad \ldots(4)$$

for $2m(k) > 2n(k) > 2k$. Let, for each even integer $2k$, $2m(k)$ be the least even integer exceeding $2n(k)$ satisfying (4), so that

$$d(Tx_{2n(k)}, Tx_{2m(k)-2}) \leqslant \epsilon,\ d(Tx_{2n(k)}, Tx_{2m(k)}) > \epsilon. \qquad \ldots(5)$$

Then, for each even integer $2k$,

$$\epsilon < d(Tx_{2n(k)}, Tx_{2m(k)}) \leqslant d(Tx_{2n(k)}, Tx_{2m(k)-2}) + d_{2m(k)-2} + d_{2m(k)-1}.$$

It follows from (3) and (5) that

$$\lim_{k\to\infty} d(Tx_{2n(k)}, Tx_{2m(k)}) = \epsilon. \qquad \ldots(6)$$

By triangular inequality,

$$| d(Tx_{2n(k)}, Tx_{2m(k)-1}) - d(Tx_{2n(k)}, Tx_{2m(k)}) | \leqslant d_{2m(k)-1}$$

and

$$| d(Tx_{2n(k)+1}, Tx_{2m(k)-1}) - d(Tx_{2n(k)}, Tx_{2m(k)}) | \leqslant d_{2m(k)-1} + d_{2n(k)}.$$

From (6), as $k \to \infty$,

$$d(Tx_{2n(k)}, Tx_{2m(k)-1}) \to \epsilon \text{ and } d(Tx_{2n(k)+1}, Tx_{2m(k)-1}) \to \epsilon.$$

By assumption (C_2)

$$\begin{aligned} d(Tx_{2n(k)}, Tx_{2m(k)}) &\leqslant d(Tx_{2n(k)}, Tx_{2n(k)+1}) + d(Tx_{2n(k)+1}, Tx_{2m(k)}) \\ &\leqslant d_{2n(k)} + h(d(Tx_{2n(k)}, Tx_{2m(k)-1}), d_{2n(k)}, d(Tx_{2m(k)}, Tx_{2m(k)}), \\ &\qquad d(Tx_{2m(k)-1}, Tx_{2n(k)+1}), d_{2m(k)-1}). \end{aligned}$$

Since h is upper semicontinuous,

$$\epsilon \leqslant h(\epsilon, 0, \epsilon, \epsilon, 0) \leqslant g(\epsilon) < \epsilon$$

as $k \to \infty$, a contradiction. Thus $\{Tx_n\}$ is a Cauchy sequence. By the completeness of X, $\{Tx_n\}$ converges to a point x in X. It follows from (1) that $\{Ex_{2n}\}$ and $\{Fx_{2n+1}\}$ also converge to x, Since E, F and T are continuous, we have

$$E(Tx_{2n}) \to Ex, \quad F(Tx_{2n+1}) \to Fx.$$

From (C_1)

$$E(Tx_{2n}) = T(Ex_{2n}), \quad F(Tx_{2n+1}) = T(Fx_{2n+1})$$

for all $n = 0, 1, \ldots$. Taking $n \to \infty$, we have

$$Ex = Tx = Fx \qquad \ldots(7)$$

and

$$T(Tx) = T(Ex) = E(Tx) = E(Ex) = T(Fx) = F(Tx) = F(Ex)$$
$$= E(Fx) = F(Fx). \qquad ...(8)$$

By (C_2), (7) and (8), if $Ex \neq F(Ex)$, then

$$d(Ex, F(Ex)) \leqslant h(d(Tx, T(Ex), d(Tx, Ex), d(Tx, F(Ex), d(T(Ex), Ex), d(T(Ex), F(Ex))$$
$$\leqslant h(d(Ex, F(Ex)), 0, d(Ex, F(Ex)), d(Ex, F(Ex)), 0)$$
$$\leqslant g(d(Ex, F(Ex)) < d(Ex, F(Ex)),$$

a contradiction. Hence $Ex = F(Ex)$. This and (8) imply Ex is a common fixed point of E, F and T.

Let u and $v (u \neq v)$ be two points of X such that $Eu = Fu = Tu = u$ and $Ev = Fv = Tv = v$. Then, by (C_2) and (C_3),

$$d(u, v) = d(Eu, Fv) \leqslant h(d(Tu, Tv), d(Tu, Eu), d(Tu, Fv), d(Tv, Eu), d(Tv, Fv))$$
$$= h(d(u, v), 0, d(u, v), d(u, v), 0)$$
$$\leqslant g(d(u, v)) < d(u, v)$$

a contradiction. Hence $u = v$. Therefore our proof is complete.

Corollary 1 — Let A be a family of continuous selfmappings of a complete metric space (X, d). Suppose there is a T in A such that, to each pair, E, F in A, the conditions (C_1) and (C_2) hold for all x, y in X. Then each S in A has a unique fixed point which is a unique common fixed point for the family A.

Corollary 2 — Let X be a metric space with two metrics d and δ. Suppose that E, F and T are selfmappings of X satisfying the conditions (C_1) and (C_2). If

(i) X is complete with respect to δ;

(ii) there exists $f: R^+ \to R^+$, continuous at 0, $f(0) = 0$ and $\delta(x, y) \leqslant f(d(x, y))$ for all x, y in X;

(iii) E, F and T are continuous with respect to δ,

then E, F and T have a unique common fixed point.

PROOF: As in the proof of Theorem 1, we see that $\{d_n\}$ is decreasing and $\{Tx_n\}$ is a Cauchy sequence with respect to d. It follows from (ii) that $\{Tx_n\}$ is a Cauchy sequence with respect to δ. From (i), $\{Tx_n\}$ has a limit x in X, i.e., $Tx_n \xrightarrow{\delta} x$. Employing the method as described in the proof of Theorem 1, we can prove E, F and T have a unique common fixed point.

Remark : Letting $f(x) = x$, $h(x, y, z, u, v) = ax + b(y + z) + c(u + v)$, where $a, b, c \in [0, 1)$ and $a + 2b + 2c < 1$ in our Corollary 2, we obtain Iseki's (1975) result.

Theorem 2 — Let E, F and T be three continuous selfmappings of a complete metric space (X, d) satisfying the condition (C_1) and the following condition :

there are two positive integers m and n such that

$$d(E^m x, F^n y) \leqslant h(d(Tx, Ty), d(Tx, E^m x), d(Tx, F^n y), d(Ty, E^m x), d(Ty, F^n y))$$

where h in H with the property (C_3).

Then E, F and T have a unique common fixed point.

PROOF : It follows from (C_1) that $E^m T = TE^m$, $F^n T = TF^n$, $E^m(X) \subset E(X) \subset T(X)$ and $F^n(X) \subset F(X) \subset T(X)$. Thus, by Theorem 1, there is a unique point x in X such that

$$x = Tx = E^m x = F^n x.$$

Also

$$T(Ex) = E(Tx) = Ex = E(E^m x) = E^m(Ex).$$

This means that Ex is a common fixed point of T and E^m. Similarly, Fx is a common fixed point of T and F^n. The uniqueness of x implies

$$Ex = Fx = Tx = x.$$

This completes our proof.

Theorem 3 — Let

(a) $T_i (i = 0, 1, ..., k)$ be continuous selfmappings of a complete metric space (X, d) with $T_i T_j = T_j T_i (i, j = 0, 1, ..., k)$;

(b) there is an h in H with the property (C_3) and two systems of positive integers $m_1, ..., m_k$ and $n_1, ..., n_k$ such that (C_1) and (C_2) hold for $E = T_1^{m_1} \dots T_k^{m_k}$, $F = T_1^{n_1} \dots T_k^{n_k}$ and $T = T_0$.

Then T_i $(i = 0, 1, ..., k)$ have a unique common fixed point.

PROOF : It follows from Theorem 1 that E, F and T have a unique common fixed point x in X, i.e.,

$$Ex = Fx = Tx = x.$$

Thus, for each i,

$$T_i(Ex) = T_i(Fx) = T_i(Tx) = T_ix.$$

From (a)

$$E(T_ix) = F(T_ix) = T(T_ix) = T_ix.$$

Hence T_ix, $i = 0, 1, \ldots, k$, are common fixed points of E, F and T. By the uniqueness of the common fixed points of E, F and T we have

$$T_ix = x \ (i = 0, 1, \ldots, k).$$

Let y and z be two common fixed points of $T_i (i = 0, 1, \ldots, k)$. Since the common fixed points of $T_i (i = 0, 1, \ldots, k)$ are also fixed points of E, F and T, by Theorem 1, we see easily that $y = z$. Thus our proof is complete.

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