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# SOLUTION OF NONLINEAR DIFFERENTIAL EQUATIONS BY USING VOLTERRA SERIES

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A technique similar to Laplace transform is extended to the solution of nonlinear differential equation. The non linear equation is first transformed into the frequency domain from the time domain by introducing the concept of multidimensional transform. By associating the variables transformation from multidimension to single dimension is obtained. Next by treating the system to be linear time domain analysis is obtained from the standard Laplace transform table.

## INTRODUCTION

This paper considers a nonlinear differential equation of the form

$$g\left(\frac{d}{dt}\right)y + \sum_{i=1}^{n} a_i y^i = x(t). \qquad \ldots(1)$$

Equation (1) can be written in the term

$$f(y(t), \dot{y}(t), \ddot{y}(t) \ldots) = x(t)$$

or $\quad K[y(t)] = x(t)$

or $\quad y(t) = H[x(t)]$

$K$ is a nonlinear operator (Waddington and Fallside 1966) and $H$ is the inverse of $K$.

## VOLTERRA FUNCTIONAL

By the method of functional expansion (Volterra 1959) it can be shown that the nonlinear system is Fig. 1. Represented by eqn. (1) can be broken down into a sequence of components connected in parallel (Weiner 1959) as shown in Fig. 2. The first component is a linear one. It is well known (Dorf 1967) that the output $y_1(t)$ is given by

$$y_1(t) = \int_0^t h_1(\tau_1)\, x(t - \tau_1)\, d\tau_1 \qquad \ldots(2)$$

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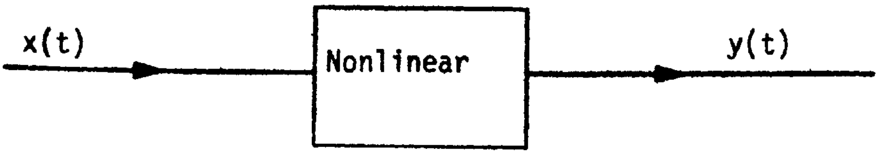


FIG. 1. Nonlinear system represented by eqn. (1).

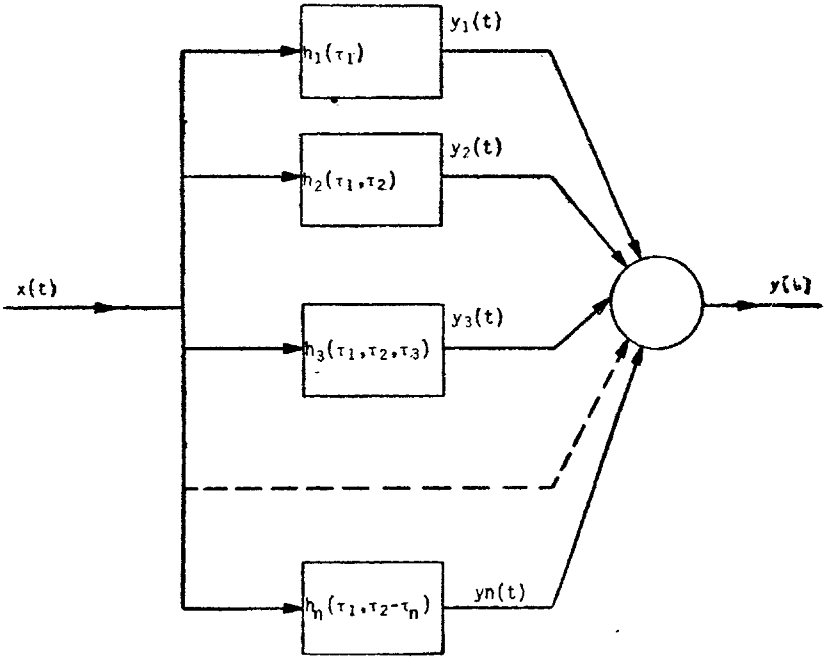


FIG. 2. Functional expansion of the nonlinear system represented by eqn. (1).

The second component is of quadratic in nature.

$$y_2(t) = \int_0^t \int_0^t h_2(\tau_1, \tau_2)\, x(t-\tau_1)\, x(t-\tau_2)\, d\tau_1 d\tau_2. \qquad \ldots(3)$$

The third component is cubic in nature and its output is

$$y_3(t) = \int_0^t \int_0^t \int_0^t h_3(\tau_1, \tau_2, \tau_3)\, x(t-\tau_1)\, x(t-\tau_2)\, n(t-\tau_3)\, d\tau_1 d\tau_2 d\tau_3. \qquad \ldots(4)$$

Hence the total output of the system is given by

$$y(t) = \sum_{i=1}^{\infty} y_i(t). \qquad \ldots(5)$$

We shall confine our discussion for the finite number of terms, and the $n$th order approximation will be given by

$$y(t) = \sum_{i=1}^{n} y_i(t) \qquad \ldots(6)$$

The various impulse responses are called Volterra kernels (Van Trees 1964)

## MULTIDIMENSIONAL TRANSFORM

With reference to Fig. 2 by the well-known relationship (Dorf 1967) of Laplace transform

$$Y_1(s_1) = H_1(s_1)\, X(s_1).$$

For the quadratic expression of (3) by letting $t_1 = t_2 = t$

$$y_2(t_1, t_2) = \int_0^{t_1}\int_0^{t_2} h_2(\tau_1, \tau_2)\, x(t_1 - \tau_1)\, x(t_2 - \tau_2)\, d\tau_1 d\tau_2.$$

By taking transform (Von Trees 1964) of both sides we get

$$Y_2(s_1, s_2) = H_2(s_1, s_2)\, X(s_1)\, X(s_2).$$

Similarly we get

$$Y_3(s_1, s_2, s_3) = H_3(s_1, s_2, s_3)\, X(s_1)\, X(s_2)\, X(s_3).$$

Hence from (6)

$$Y(s_1, s_2, \ldots, s_n) = \sum_{i=1}^{n} H_i(s_1, s_2, \ldots, s_i) \prod_{i=1}^{n} X(s_i). \qquad \ldots(7)$$

Where $H_n(s_1, s_2, \ldots, s_n)$ is defined to be $n$-dimensional transform (Kuo 1977) and is given by

$$H_n(s_1, s_1, \ldots, s_n) = \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} h_n(t_1, t_2, \ldots, t_n) \exp(s_1 t_1 + s_n t_n)\, dt_1, \ldots, dt_n. \qquad \ldots(8)$$

Physically we may interpret $H$ as follows : $H_1(s_1)$ produces a single frequency response to a single input $\exp(s_1 t)$; $H_2(s_1, s_2)$ produces a double frequency response to an input $\exp(s_1 + s_2)\, t$; and $H_n(s_1, s_2, \ldots, s_n)$ accounts for the system as sum frequencies $\exp(s_1 + s_2, \ldots, s_n)\, t$ input.

Now if $H_1(s_1)$, $H_2(s_1, s_2)$ etc. are known, we can from (7) determine $Y(s_1, s_2, \ldots, s_n)$ in term of $X(s_1)$, $X(s_2)$, $X(s_n)$ etc.

By the technique of associated transform (Lubbock and Bansal 1969, Flake 1963) eqn. (7) can be expressed in term of only one frequency $s$

Next step is to find $y(t)$ from $Y(s)$ by inverse Laplace transform (Nixon 1965).

## Computation of Volterra Transform Function

In the mathematical literature the quantity $H_n(s_1, s_2, \ldots, s_n)$ is interchangably termed as Volterra transfer function, multidimensional transform or nonlinear transfer function, considerable work has been done (Van Trees 1964, Bedrosian and Rice 1971) on the determination of the Volterra transfer function when the system equation is known. One common method is the Harmonic Input Method. This method is based on the fact that a harmonic input to a nonlinear system will produce a harmonic output.

Thus if $x(t) = \exp(s_1 + s_2, \ldots, + s_n)t$,

$H_n(s_1, s_2, \ldots, s_n)$ will be given by the coefficient of the $\exp(s_1 + s_2 +, \ldots, s_n)t$ term in $y(t)$ of (6).

Bedrosian and Rice (1971) have given a general expression of $H_n(s_1, s_2, \ldots, s_n)$ for the equation of the form (1). Following their method of calculation the first three tranfer functions are given below.

$$H_1(s_1) = \frac{1}{g(s_1) + a_1} \qquad \ldots(9)$$

$$H_2(s_1, s_2) = -\frac{a_2 H_1(s_1)\, H_1(s_2)}{g(s_1 + s_2) + a_1}. \qquad \ldots(10)$$

$$H_3(s_1, s_2, s_3) = -\frac{a_2\, \Sigma'_3\, H_1(s_1)\, H_2(s_1, s_2) + a_3 H_1(s_1)\, H_1(s_2)\, H_1(s_3)}{g(s_1 + s_2 + s_3) + a_1} \qquad \ldots(11)$$

## Example

We shall clarify our above discussion with the help of an example. Let us consider the nonlinear equation

$$\dot{y} + \alpha y^2 = \beta t$$

$$\alpha, \beta \neq 0, \dot{y}(0) = y(0) = 0$$

By using the second order approximation from (7)

$$Y(s_1, s_2) = H(s_1)\, X(s_1) + H_2(s_1, s_2)\, X(s_1)\, X(s_2).$$

From (9) and (10) we get

$$H_1(s_1) = \frac{1}{s_1},\; H_2(s_1, s_2) = -\alpha H_1(s_1)\, H(s_2)\, H_1(s_1 + s_2)$$

$$Y(s_1, s_2) = \frac{1}{s_1} \cdot \frac{\beta}{s_i^2} - \alpha \cdot \frac{1}{s_1} \cdot \frac{1}{s_2} \cdot \frac{1}{s_1 + s_2} \cdot \frac{\beta}{s_1^2} \cdot \frac{\beta}{s_2^2}$$

$$Y(s_1, s_2) = \frac{\beta}{s_1^3} - \alpha\beta^2 \cdot \frac{1}{s_1^3\, s_2^3\, (s_1 + s_2)}.$$

Now by associating the variables (Lubbock and Bansal 1969) we can express $Y$ in term of only one frequency $s$

$$Y(s) = \frac{\beta}{s^3} - \alpha\beta^2 \frac{6}{s^6}.$$

$$\text{Thus,} \quad y(t) = \frac{\beta t^2}{2}\left(1 - \frac{\alpha\beta t^3}{10}\right).$$

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