Indian J. pure appl. Math., **10**(4) : 426-430, April 1979

FIXED POINT THEOREMS UNDER GENERALIZED CONTRACTIVE MAPPINGS

HARUO MURAKAMI

Department of Applied Mathematics, Kobe University, Japan

AND

CHEH-CHIH YEH

Department of Mathematics, Central University, Chung-Li, Taiwan 320

(*Received* 9 *October* 1978)

Let T be a continuous mapping of a complete metric space (X, d). Suppose that there are functions p, q, r, s, h and k of $(0, \infty)$ into $[0, 1)$ such that

(i) $p(t) + q(t) + r(t) + s(t) + h(t) + k(t) < 1$ for each $t > 0$;

(ii) p, q, r, s, h and k are upper semicontinuous from the right;

(iii) $d(T^{2m}x, T^{2m}y)\, d(T^mx, T^my) \leqslant d(T^mx, T^my)\, \{p(d(T^mx, T^my))$
$\times\, d(T^mx, T^my) + q(d(T^mx, T^my))\, d(T^mx, T^{2m}x)$
$+\, r(d(T^mx, T^my))\, d(T^my, T^{2m}y) + s(d(T^mx, T^my))\, d(T^mx, T^{2m}y)$
$+\, h(d(T^mx, T^my))\, d(T^my, T^{2m}x)\}$
$+\, k(d(T^mx, T^my))\, d(T^mx, T^{2m}x)\, d(T^my, T^{2m}y)$

for all x, y in X, $x \neq y$, where m is some positive constant.

Under these assumptions our main result states that T has a unique fixed point.

1. INTRODUCTION

Let (X, d) be a complete metric space and let $T : X \to X$. Banach's classical fixed point theorem guarantees the existence of a unique fixed point for T provided it satisfies

$$d(Tx, Ty) \leqslant cd(x, y)$$

where $0 \leqslant c < 1$ and x, y in X.

Wong (1974) proved some fixed point theorems for certain T which are controlled by five functions α_i's from $(0, \infty)$ into $[0, \infty)$ such that for any distinct x, y in X

$$d(Tx, Ty) \leqslant a_1 d(x, Tx) + a_2 d(y, Ty) + a_3 d(x, y) + a_4 d(y, Tx) + a_5 d(x, Ty)$$

where $a_i = \dfrac{\alpha_i(d(x, y))}{d(x, y)}$, and the control functions satisfy regular and contractive conditions such as the upper semicontinuity from the right for each α_i and $\sum_{i=1}^{5} \alpha_i(t) < t$ for $t > 0$.

The main purpose of this note is to prove some fixed point theorems for mappings which satisfy a condition of the type (iii). For related recent results, we refer to Hardy and Rogers (1974), Murakami and Yeh (1978) and Raktoch (1962).

2. Main Results

Theorem 1 — Let T be a continuous selfmapping of a complete metric space (X, d). Suppose that there are functions p, q, r, s, h and k of $(0, \infty)$ into $[0, 1)$ such that

(C_1) $p(t) + q(t) + r(t) + s(t) + h(t) + k(t) < 1$ for each $t > 0$;

(C_2) p, q, r, s, h and k are upper semicontinuous from the right;

(C_3)
$$\begin{aligned} & d(T^{2m}x, T^{2m}y)\, d(T^m x, T^m y) \\ & \leqslant d(T^m x, T^m y)\, \{p(d(T^m x, T^m y))\, d(T^m x, T^m y) \\ & + q(d(T^m x, T^m y))\, d(T^m x, T^{2m}x) + r(d(T^m x, T^m y))\, d(T^m y, T^{2m}y) \\ & + s(d(T^m x, T^m y))\, d(T^m x, T^{2m}y) + h(d(T^m x, T^m y))\, d(T^m y, T^{2m}x)\} \\ & + k(d(T^m x, T^m y))\, d(T^m x, T^{2m}x)\, d(T^m y, T^{2m}y) \end{aligned}$$

for x, y in X, $x \neq y$, where m is some positive constant. Then T has a unique fixed point.

Proof: Let x be any point of X. Define $x_0 = T^m x$ and

$$x_{n+1} = T^m x_n, \quad d_n = d(x_n, x_{n+1}) \qquad \ldots(1)$$

for $n = 0, 1, \ldots$.

We shall prove first that T^m has a fixed point. Suppose that $d_n > 0$ for each n, if not, then x_n is a fixed point of T^m. Hence, by (C_3)

$$\begin{aligned} d_n d_{n+1} &= d(T^{2m}x_{n-1}, T^{2m}x_n)\, d(T^m x_{n-1}, T^m x_n) \\ & \leqslant d(T^m x_{n-1}, T^m x_n)\, \{p(d(T^m x_{n-1}, T^m x_n))\, d(T^m x_{n-1}, T^m x_n) \\ & \quad + q(d(T^m x_{n-1}, T^m x_n))\, d(T^m x_{n-1}, T^{2m}x_{n-1}) \\ & \quad + r(d(T^m x_{n-1}, T^m x_n))\, d(T^m x_n, T^{2m}x_n) \\ & \quad + s(d(T^m x_{n-1}, T^m x_n))\, d(T^m x_{n-1}, T^{2m}x_n) \\ & \quad + h(d(T^m x_{n-1}, T^m x_n))\, d(T^m x_n, T^{2m}x_{n-1})\} \\ & \quad + k(d(T^m x_{n-1}, T^m x_n))\, d(T^m x_{n-1}, T^{2m}x_{n-1})\, d(T^m x_n, T^{2m}x_n) \\ & = d_n\{p(d_n)\, d_n + q(d_n)\, d_n + r(d_n)\, d_{n+1} + s(d_n)\, (d_n + d_{n+1})\} \\ & \qquad + k(d_n)\, d_n d_{n+1} \end{aligned}$$

which implies

$$(1 - r(d_n) - s(d_n) - k(d_n))\, d_{n+1} \leqslant (p(d_n) + q(d_n) + s(d_n))d_n. \qquad \ldots(2)$$

By the symmetry property of metric, we have

$$(1 - q(d_n) - h(d_n) - k(d_n))\, d_{n+1} \leqslant (p(d_n) + r(d_n) + h(d_n))\, d_n. \qquad \ldots(3)$$

Adding (2) and (3) and using (C_1) we have

$$d_{n+1} \leqslant \frac{2p(d_n) + q(d_n) + r(d_n) + s(d_n) + h(d_n)}{2 - 2k(d_n) - q(d_n) - r(d_n) - s(d_n) - h(d_n)}\, d_n \qquad \ldots(4)$$

$$\equiv A(d_n)\, d_n < d_n.$$

So $\{d_n\}$ is decreasing and therefore converges to some point u in $[0, \infty)$. If $u > 0$, then

$$u = \lim_{n\to\infty} d_{n+1} \leqslant \limsup_{n\to\infty} A(d_n)\, d_n. \qquad \ldots(5)$$

It follows from (C_2) and (5) that $u \leqslant A(u)\, u$, a contradiction to (C_1). Hence $u = 0$.

Next, we show that $\{x_n\}$ is a Cauchy sequence. Assume to the contrary that there are a positive number c and two sequences $\{H(n)\}$, $\{K(n)\}$ such that for each $n = 0, 1, \ldots$

$$H(n) > K(n) > n,\ d(x_{H(n)}, x_{K(n)}) \geqslant c \qquad \ldots(6)$$

and (by the well-ordering principle)

$$d(x_{H(n)-1}, x_{K(n)}) < c. \qquad \ldots(7)$$

Let $t_{n+i} = d(x_{H(n)+i}, x_{K(n)+i})$ for $n, i = 0, 1, \ldots$. Then

$$c \leqslant t_n \leqslant d(x_{H(n)}, x_{H(n)-1}) + d(x_{H(n)-1}, x_{K(n)}) < d_{H(n)-1} + c.$$

Since $\{d_n\}$ converges to zero, $\{t_n\}$ converges to c from the right. From (C_3)

$$\begin{aligned} t_n t_{n+1} &= d(T^m x_{H(n)-1}, T^m x_{K(n)-1})\, d(T^{2m} x_{H(n)-1}, T^{2m} x_{K(n)-1}) \\ &\leqslant t_n \{p(t_n)\, t_n + q(t_n)\, d_{H(n)} + r(t_n)\, d_{K(n)} \\ &\quad + s(t_n)\, d(x_{H(n)}, x_{K(n)+1}) + h(t_n)\, d(x_{K(n)}, x_{H(n)+1}\} \\ &\quad + k(t_n)\, d_{H(n)} d_{K(n)}. \end{aligned}$$

Letting $n \to \infty$, we have

$$c^2 \leqslant c\,\{p(c)\, c + s(c)\, c + h(c)\, c\} + k(c)\, c^2$$

a contradiction to (C_1). Hence $\{x_n\}$ is a Cauchy sequence. By the completeness of X, $\{x_n\}$ converges to some point w in X. Using the continuity of d and T,

$$\lim_n d_n = \lim_n d(x_n, x_{n+1}) = \lim_n d(x_n, T^m x_n) = d(w, T^m w) = 0.$$

Thus $T^m w = w$.

If possible, let z be a another fixed point of T^m, then, by (C_3),

$$\begin{aligned} & d(T^{2m}w, T^{2m}z)\, d(T^m w, T^m z) \\ & \quad \leqslant d(T^m w, T^m z)\,\{p(d(T^m w, T^m z))\, d(T^m w, T^m z) \\ & \qquad + q(d(T^m w, T^m z))\, d(T^m w, T^{2m} w) + r(d(T^m w, T^m z))\, d(T^m z, T^{2m} z) \\ & \qquad + s(d(T^m w, T^m z))\, d(T^m w, T^{2m} z) + h(d(T^m w, T^m z)\, d(T^m z, T^{2m} w)\} \\ & \qquad + k(d(T^m w, T^m z))\, d(T^m w, T^{2m} w)\, d(T^m z, T^{2m} z) \end{aligned}$$

which implies

$$(d(w, z))^2 \leqslant (d(w, z))^2\, (p(d(w, z)) + s(d(w, z)) + h(d(w, z)))$$

a contradiction to (C_1). Hence $w = z$. Thus T^m has a unique fixed point w in X. Since

$$T^m(Tw) = T(T^m w) = Tw$$

Tw also is a fixed point of T^m. By the uniqueness of the fixed point of T^m, $Tw = w$. This proves our Theorem.

Theorem 2 — Let T be a continuous selfmapping of a complete metric space (X, d). Suppose there are functions p, q, r, s, h and k of $(0, \infty)$ into $[0, 1)$ satisfying the conditions (C_1) and (C_2) and the following condition :

$$\begin{aligned} (C_4)\quad & d(T^m x, T^n y)\, d(x, y) \leqslant d(x, y)\,\{p(d(x, y))\, d(x, y) \\ & + q(d(x, y))\, d(x, T^m x) + r(d(x, y))\, d(y, T^n y) \\ & + s(d(x, y))\, d(x, T^n y) + h(d(x, y))\, d(y, T^m x)\} \\ & + k(d(x, y))\, d(x, T^m x)\, d(y, T^n y) \end{aligned}$$

for all x, y in X, $x \neq y$, where m, n are some positive constants.

Then T has a unique fixed point in X.

PROOF : Let x_0 be any point in X and define

$$x_1 = T^m x_0,\ x_2 = T^n x_1 \text{ and in general}$$

$$x_{2n} = T^n x_{2n-1},\ x_{2n+1} = T^m x_{2n}.$$

The rest of the proof of this Theorem is similar to that of our Theorem 1 with suitable modification, so we omit the details.

Remark : Letting $m = n = 1$ and $k = 0$ in Theorem 2, we have a result of Wong (1974).

References

Hardy, G., and Rogers, T. (1974). A generalization of a fixed point theorem of Reich. *Can. Math. Bull.*, **16**, 257-59.

Murakami, H., and Yeh, C. C. (1978). Fixed point theorems in complete metric spaces. *math. Japonicae*, **23**, 77-83.

Rakotch, E. (1962). A note on contractive mappings. *Proc. Am. Math. Soc.*, **13**, 459-65.

Wong, C. S. (1974). Generalized contractions and fixed point theorems. *Proc. Am. math. Soc.*, **42**, 409-417.